Decision Diagrams for Discrete Optimization

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Carnegie Mellon University

ICAPS Tutorial
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Companion Tutorial:

Decision Diagrams
for Sequencing and Scheduling

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Carnegie Mellon University

Immediately follows this tutorial

Decision Diagrams

- Used in **computer science** and **AI** for decades
  - Logic circuit design
  - Product configuration
- A **new perspective** on optimization
  - Constraint programming
  - **Discrete optimization**

Decision Diagrams

- Relevance to **planning and scheduling**:
  - Naturally suited to **dynamic programming** formulations.
    - State-dependent actions and costs.
  - New method for defeating **curse of dimensionality**.
    - Branch-and-bound solution.

Decision Diagrams

- DDs have been used in **planning literature**…
  - To encode (or approximate) state-dependent **cost** or **cost-to-go**.
  - See yesterday's **tutorial** "Planning with State-Dependent Action Costs"
    - By Robert Mattmüller and Florian Geißer.
- This tutorial presents DDs as a **general optimization method**.

Decision Diagrams

- General advantages:
  - No need for **inequality** formulations.
  - No need for **linear** or **convex** relaxations.
  - New approach to solving **dynamic programming** models.
  - Very effective **parallel** computation.
  - Ideal for **postoptimality** anaylsis
- Disadvantage:
  - Developed only for **discrete**, **deterministic** optimization.
  - …so far.

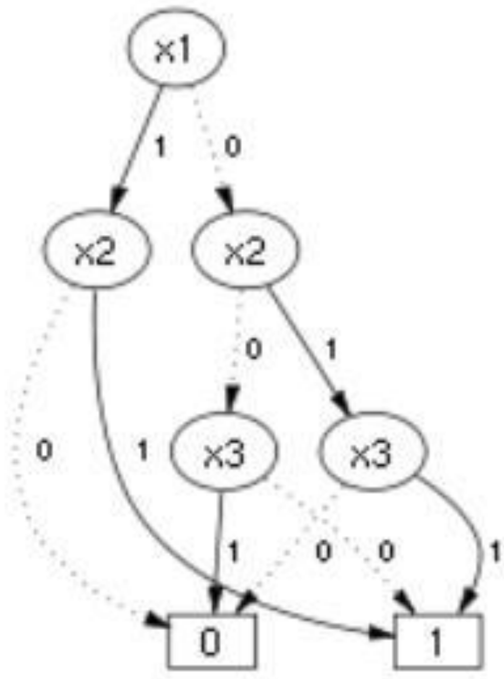
Outline

- Decision diagram **basics**
- Optimization with **exact** decision diagrams
- A **general-purpose** solver that scales up
  - **Relaxed** decision diagrams
  - **Restricted** decision diagrams
  - **Dynamic programming** model
  - A new **branching** algorithm
  - Computational **performance**
- Modeling the **objective function**
  - Inventory management example
- **Nonserial** decision diagrams
- References

Decision Diagram Basics

- Binary decision diagrams encode Boolean functions

x_1	x_2	x_3	f
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1



Lee (1959), Akers (1978)

Decision Diagram Basics

- Binary decision diagrams encode Boolean functions
  - Historically used for circuit design & verification

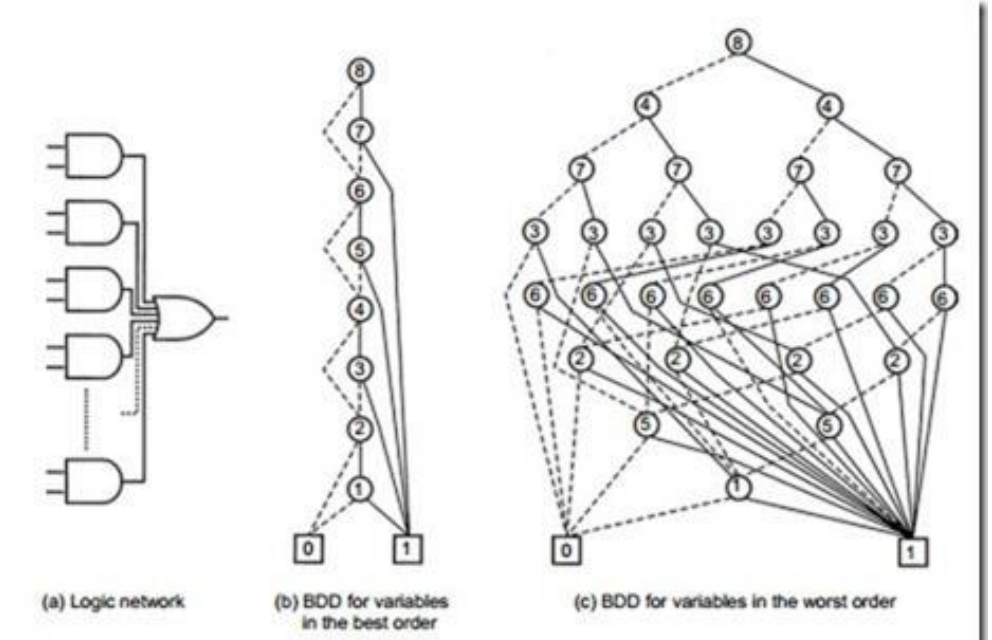
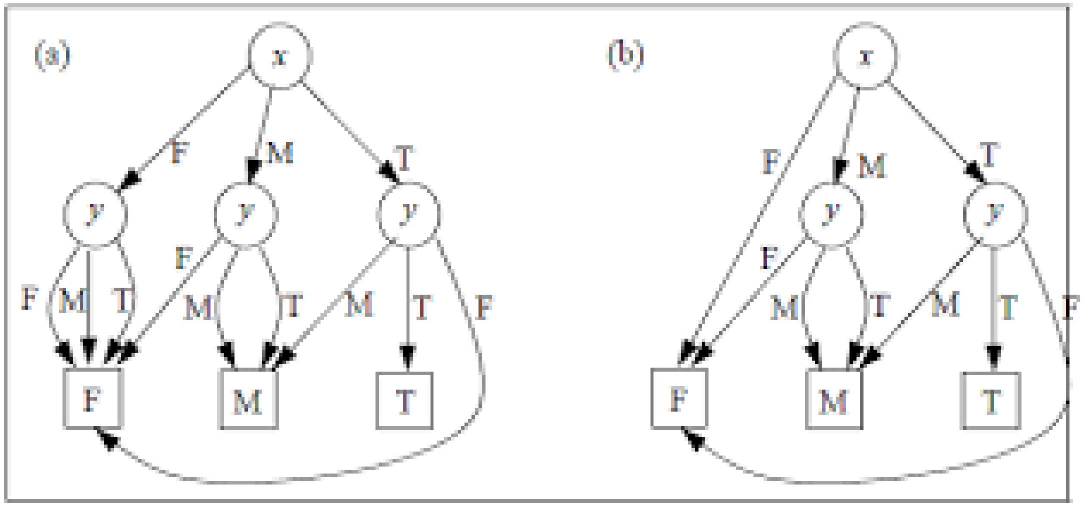


FIGURE 29.11 BDDs for 2-level logic network with AND/OR gates.

Bryant (1986), etc.

Decision Diagram Basics

- Binary decision diagrams encode Boolean functions
  - Historically used for circuit design & verification
  - Easily generalized to multivalued decision diagrams

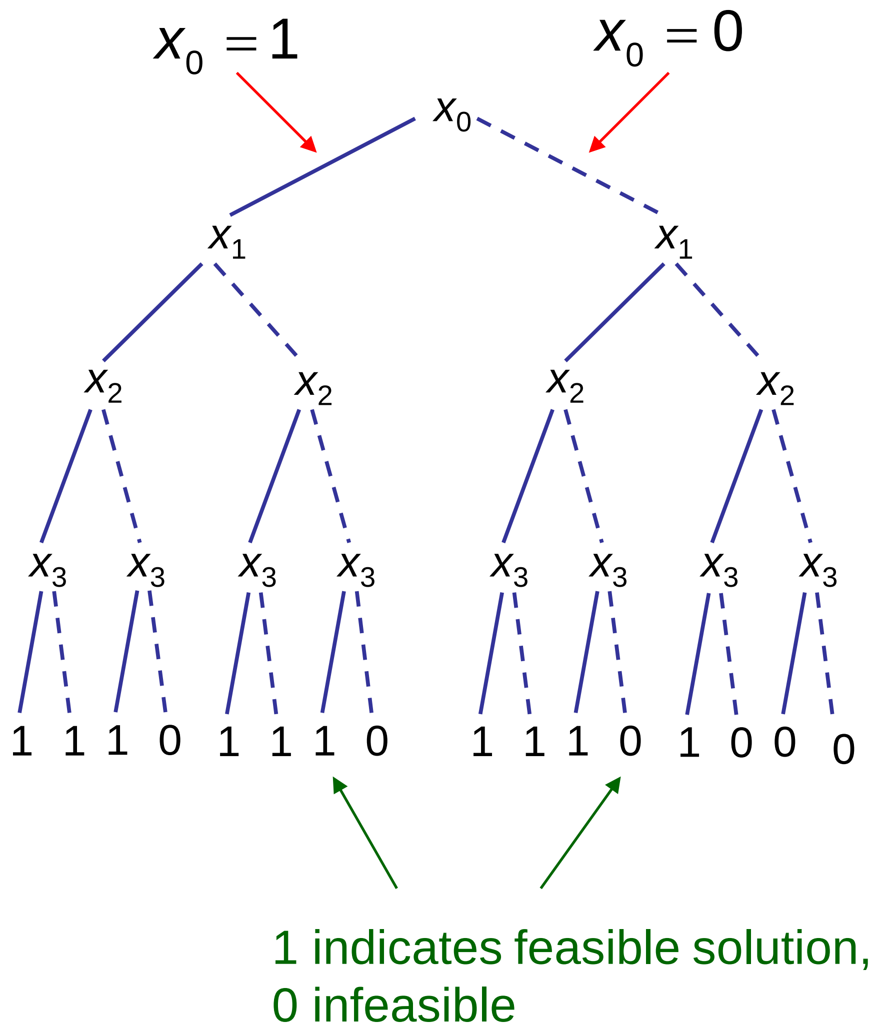


Reduced Decision Diagrams

- There is a **unique reduced** DD representing any given Boolean function.
  - Once the variable ordering is specified.

Bryant (1986)

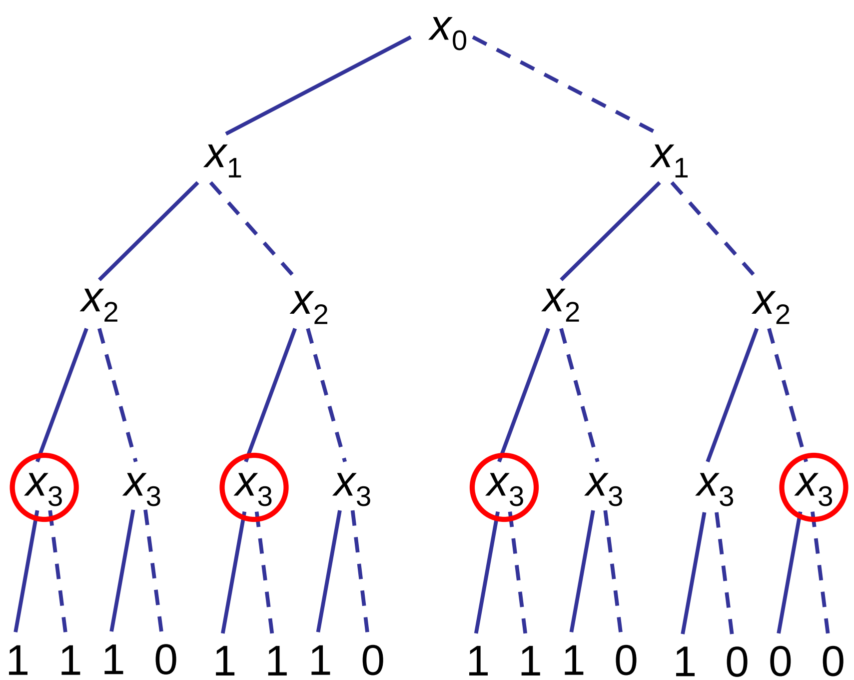
- The reduced DD can be viewed as a branching tree with **redundancy** removed.
  - Superimpose isomorphic subtrees.
  - Remove redundant nodes.



Branching tree for 0-1 inequality

$$2x_0 + 3x_1 + 5x_2 + 5x_3 \geq 7$$

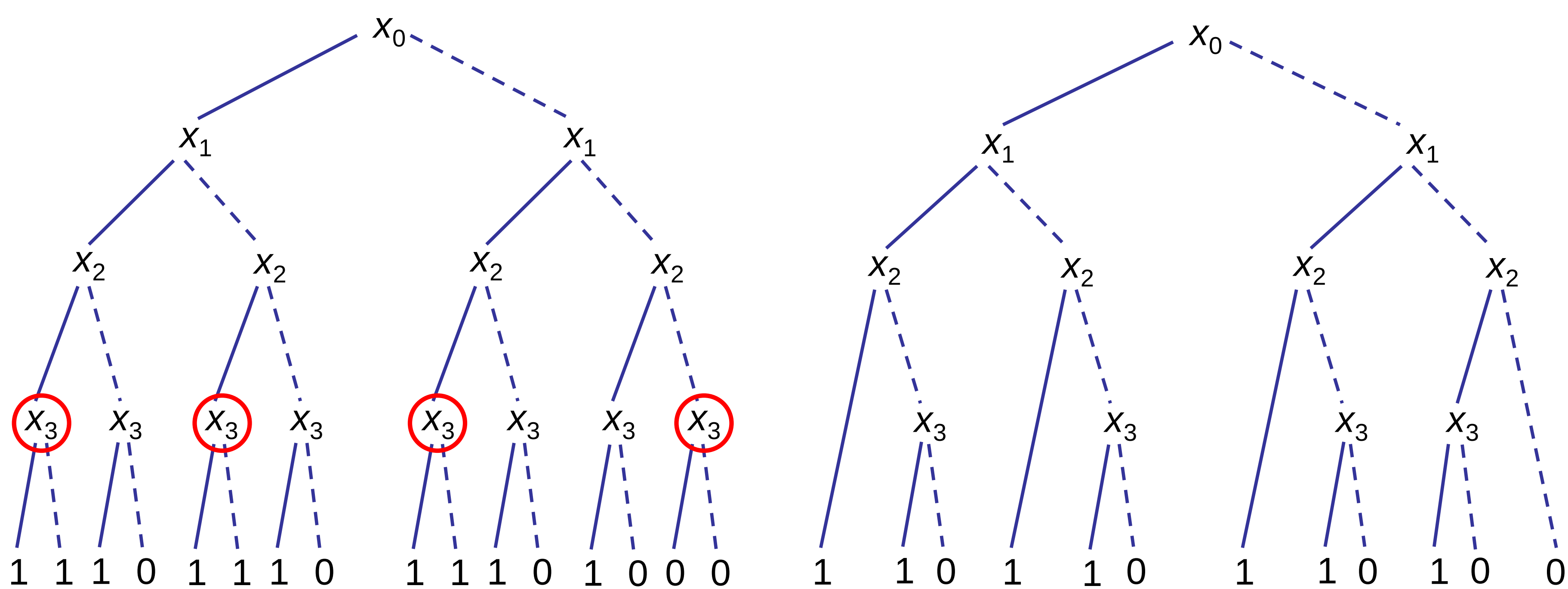
1



Branching tree for 0-1 inequality

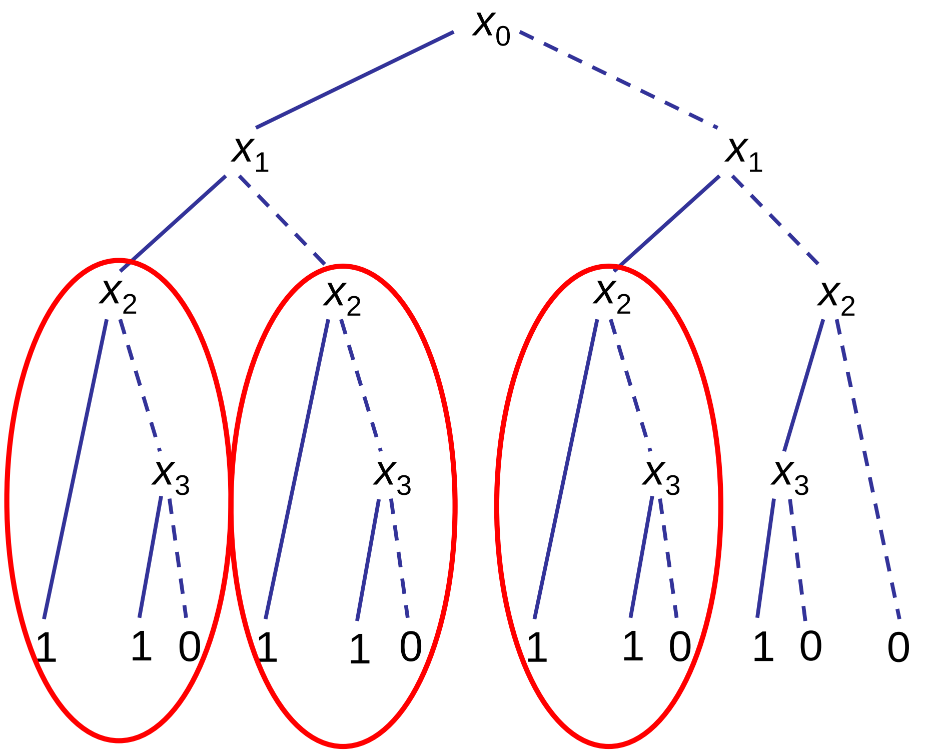
$$2x_0 + 3x_1 + 5x_2 + 5x_3 \geq 7$$

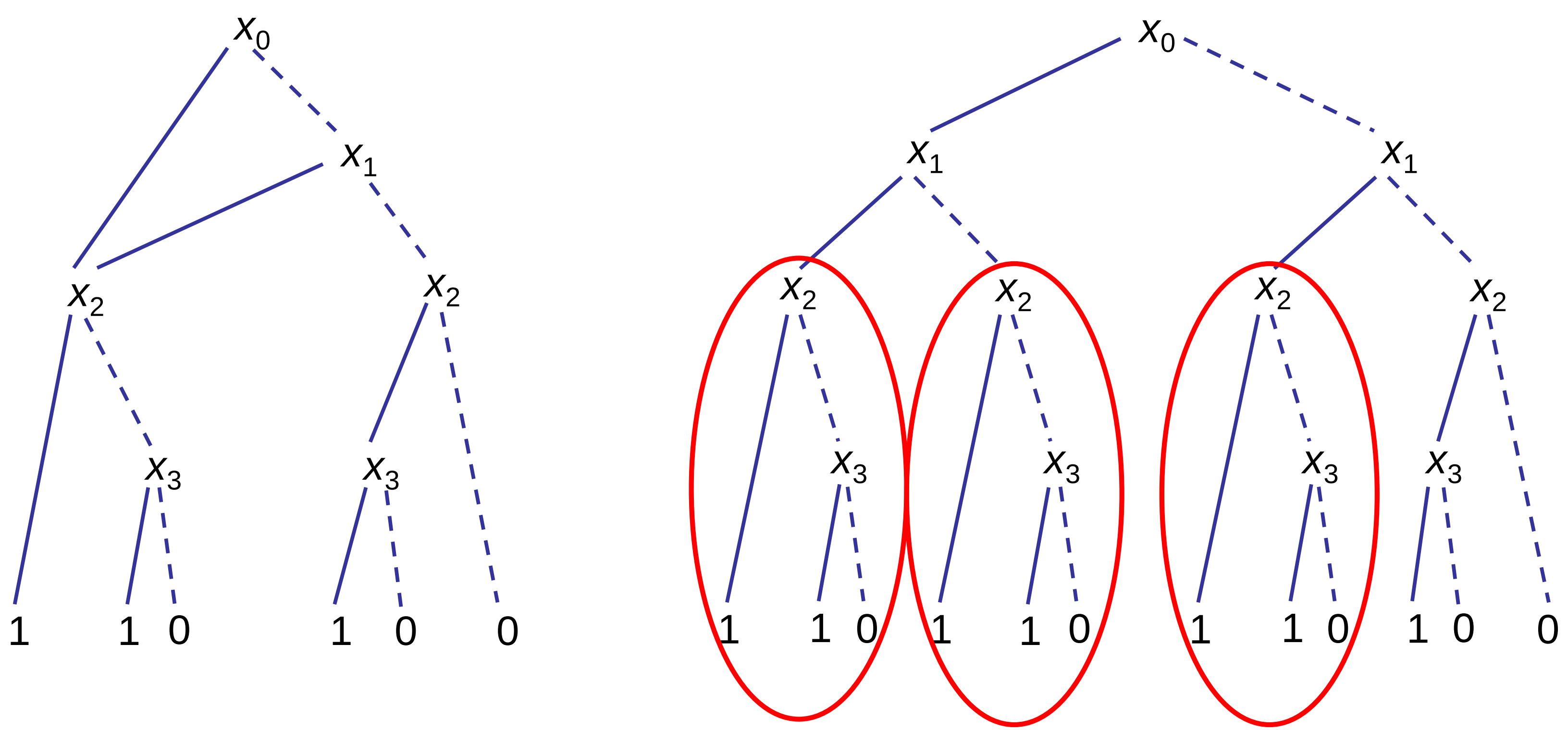
Remove redundant nodes…



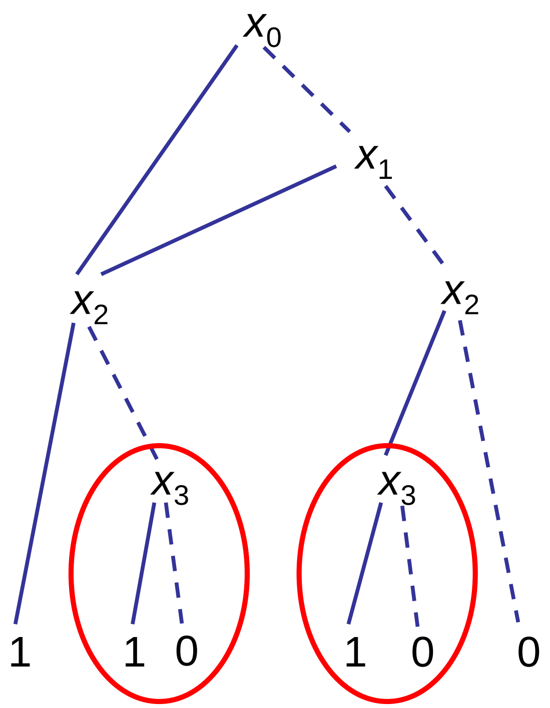
x_0
x_1 x_1
x_2 x_2 x_2 x_2
x_3 x_3 x_3 x_3 x_3 x_3 x_3 x_3
1 1 1 0 1 1 1 0 1 1 1 0 1 0 0 0
x_0
x_1 x_1
x_2 x_2 x_2 x_2
x_3 x_3 x_3 x_3
1 1 0 1 1 0 1 1 0 1 0 0

Superimpose identical subtrees…

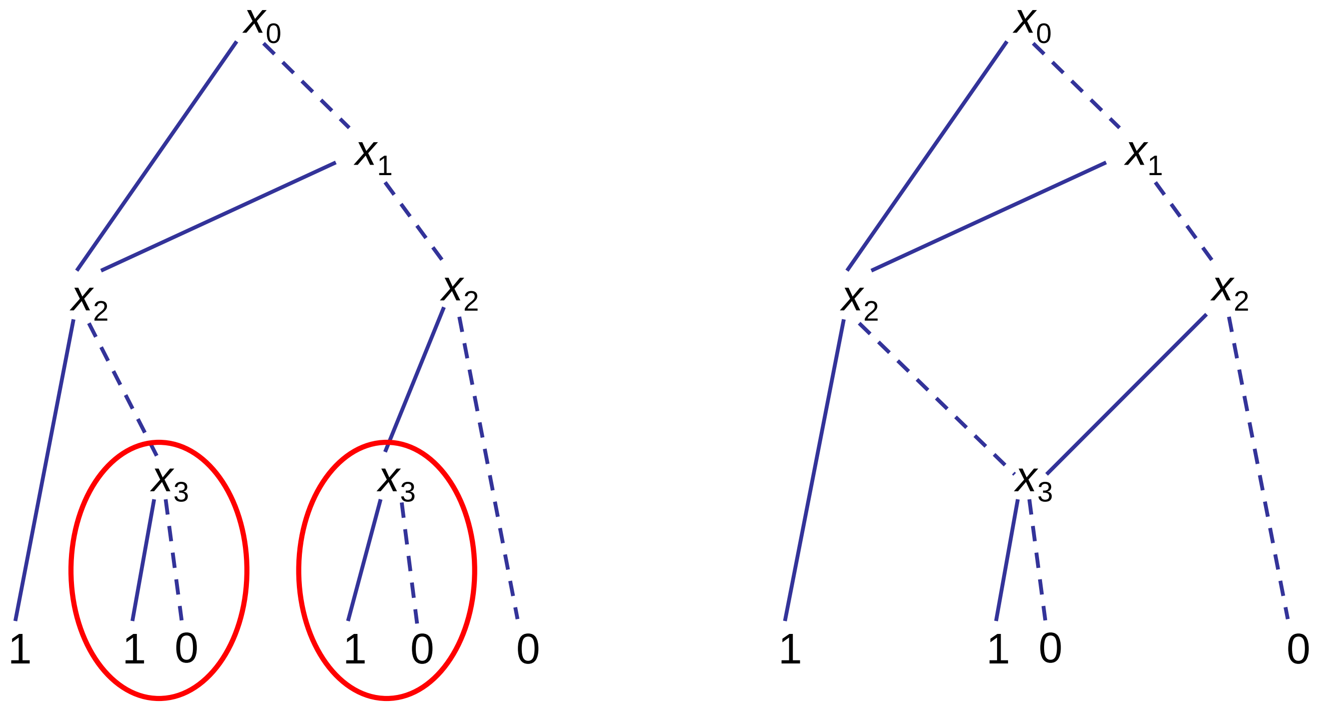




x_0
x_1
x_2
x_2
x_3
x_3
1 1 0 1 0 0
x_0
x_1
x_1
x_2
x_2
x_2
x_2
x_3
x_3
x_3
x_3
1 1 0 1 1 0 1 1 0 1 0 0

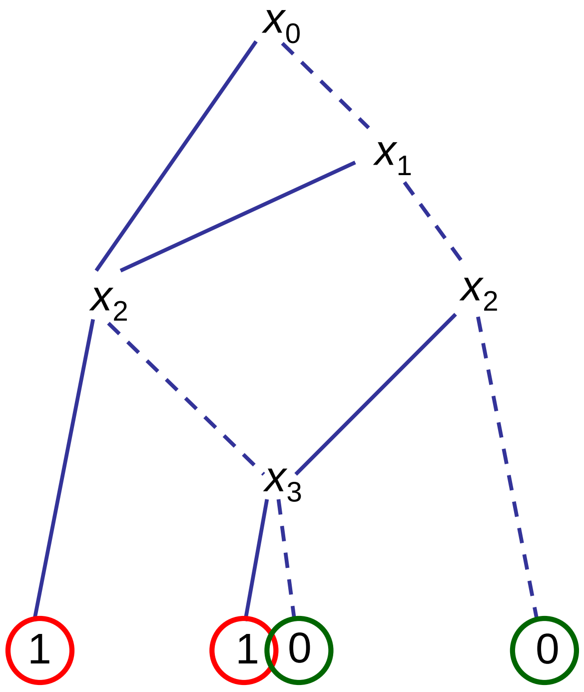


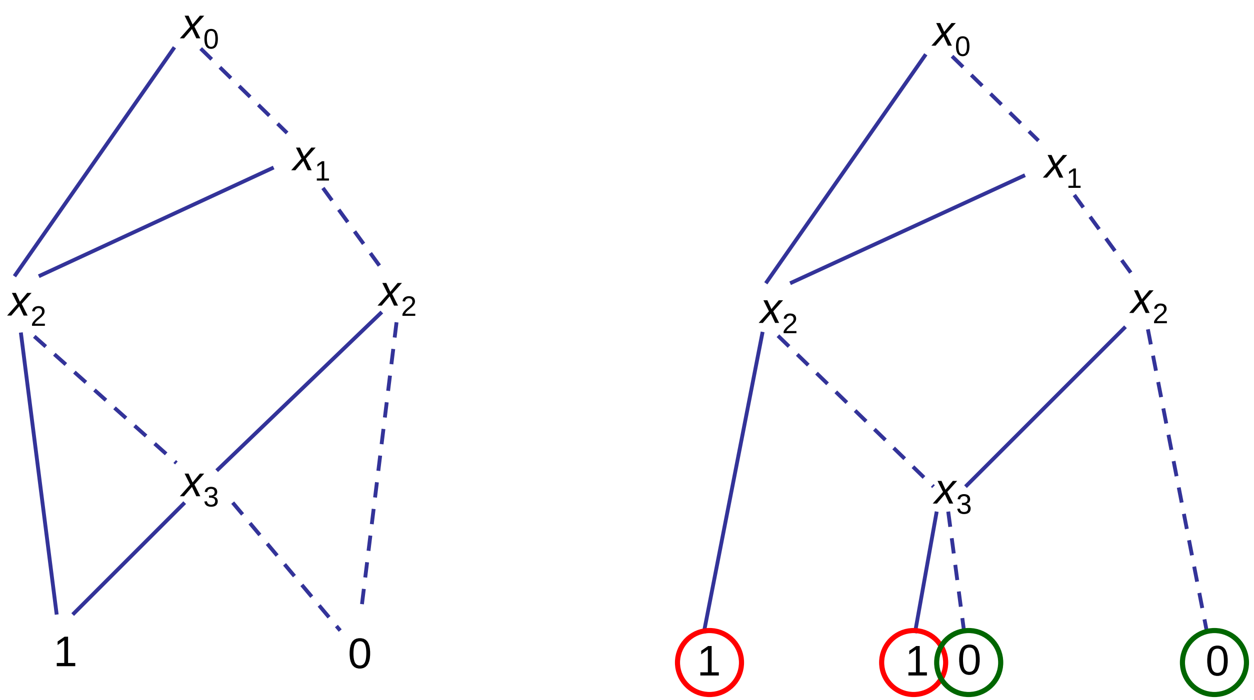
Superimpose identical subtrees…



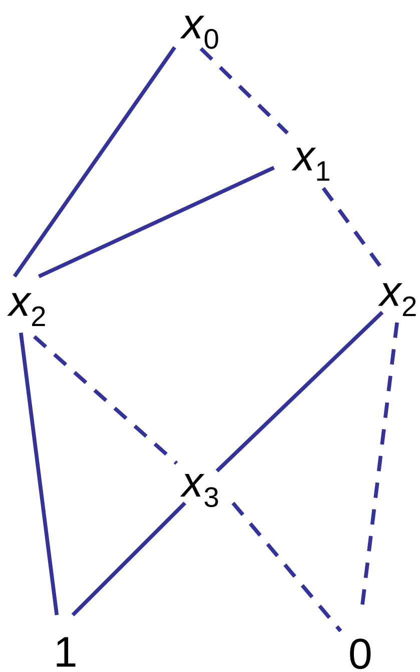
x_0
x_1
x_2
x_2
x_3
x_3
1
1
0
1
0
0
x_0
x_1
x_2
x_2
x_3
1
1
0
0

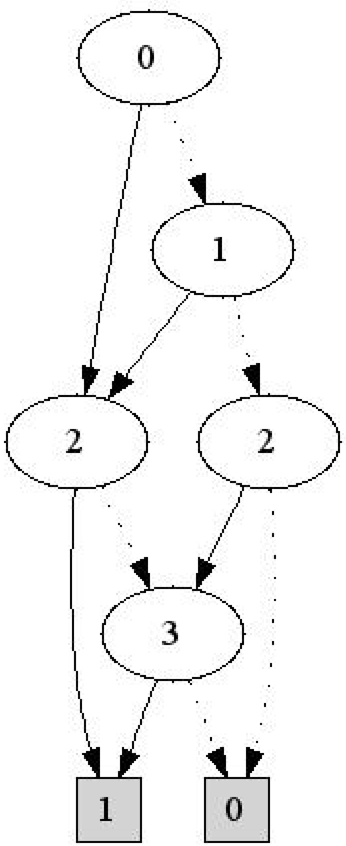
Superimpose identical leaf nodes…





x_0
x_1
x_2
x_2
x_3
1
0
x_0
x_1
x_2
x_2
x_3
1
1
0
0





as generated by software

Reduced Decision Diagrams

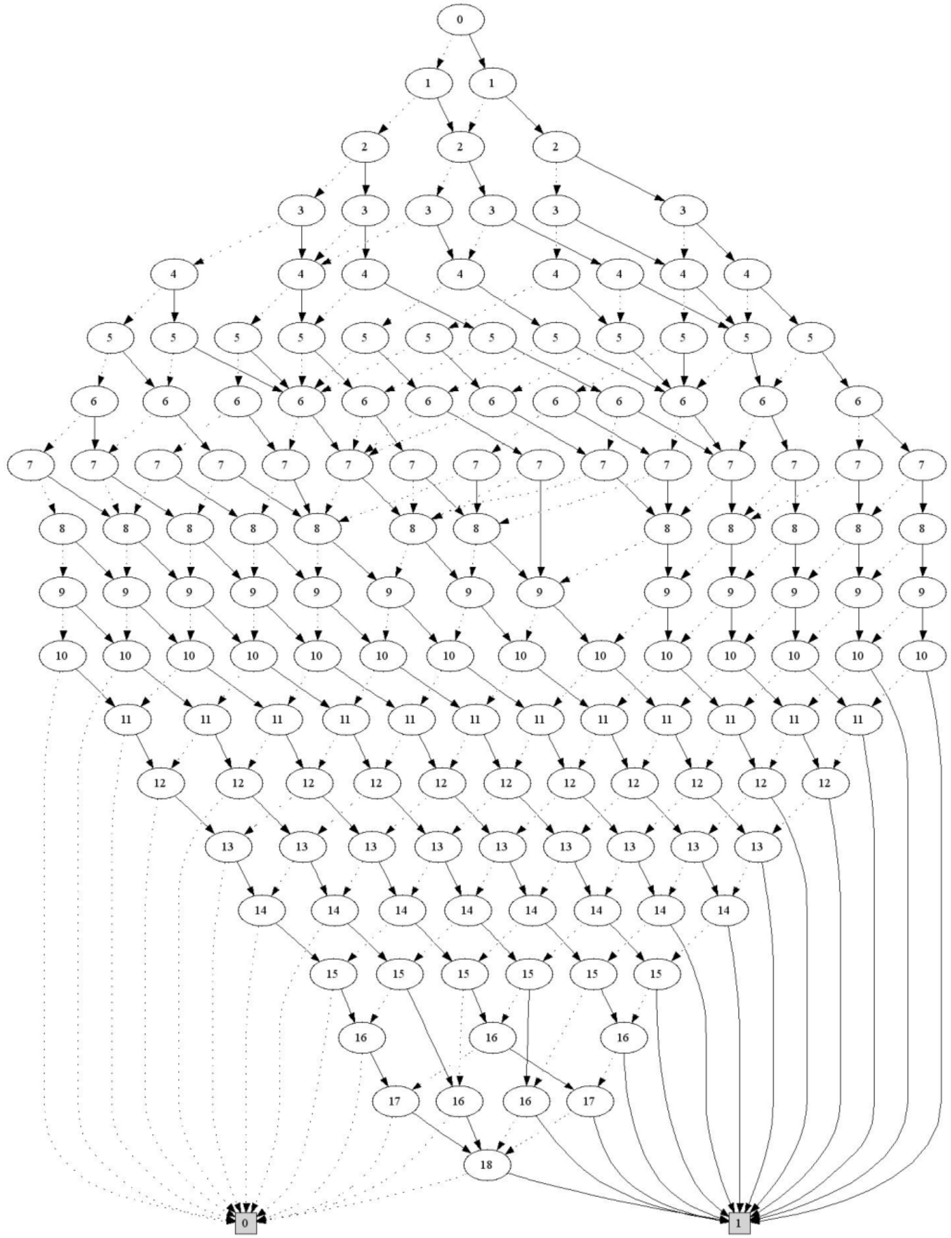
- Reduced DD for a knapsack constraint can be surprisingly small…

The 0-1 inequality

$$300x_0 + 300x_1 + 285x_2 + 285x_3 + 265x_4 + 265x_5 + 230x_6 + 230x_7 + 190x_8 + 200x_9 +$$
$$400x_{10} + 200x_{11} + 400x_{12} + 200x_{13} + 400x_{14} + 200x_{15} + 400x_{16} + 200x_{17} + 400x_{18} \leq 2700$$

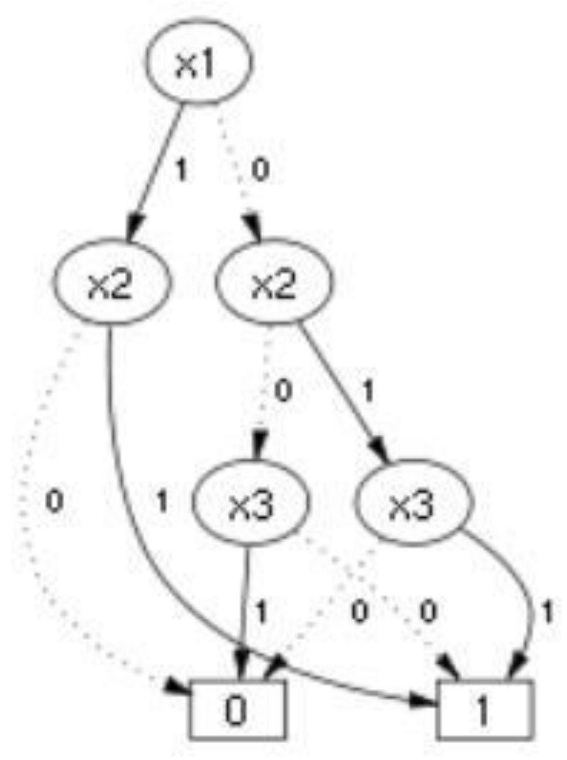
has 117,520 maximal feasible solutions (or minimal covers)

But its reduced BDD has only 152 nodes…



Optimization with Exact Decision Diagrams

- Decision diagrams can represent feasible set
  - Remove paths to 0.
  - Paths to 1 are feasible solutions.
  - Associate costs with arcs.
  - Find longest/shortest path

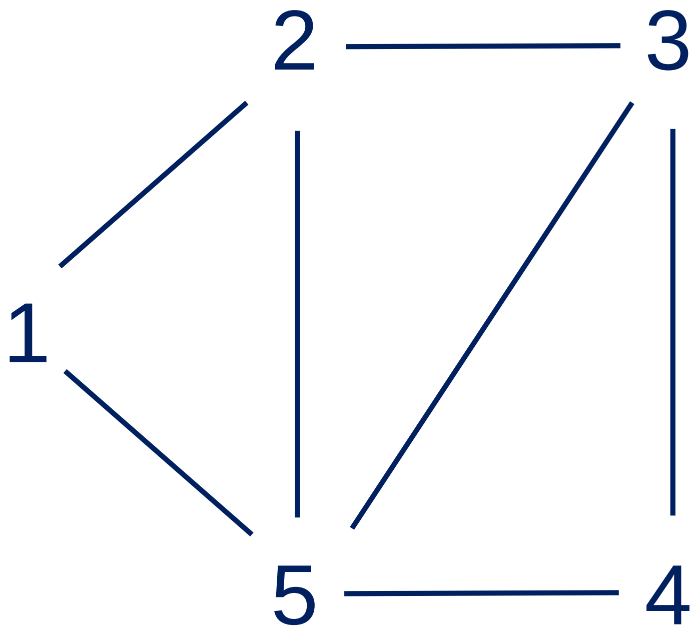


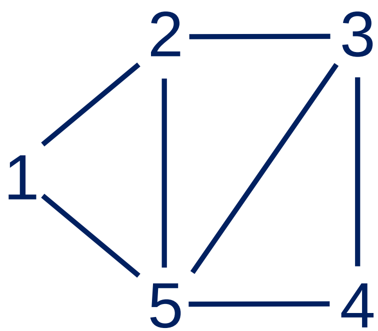
Hadžić and JH (2006, 2007)

Stable Set Problem

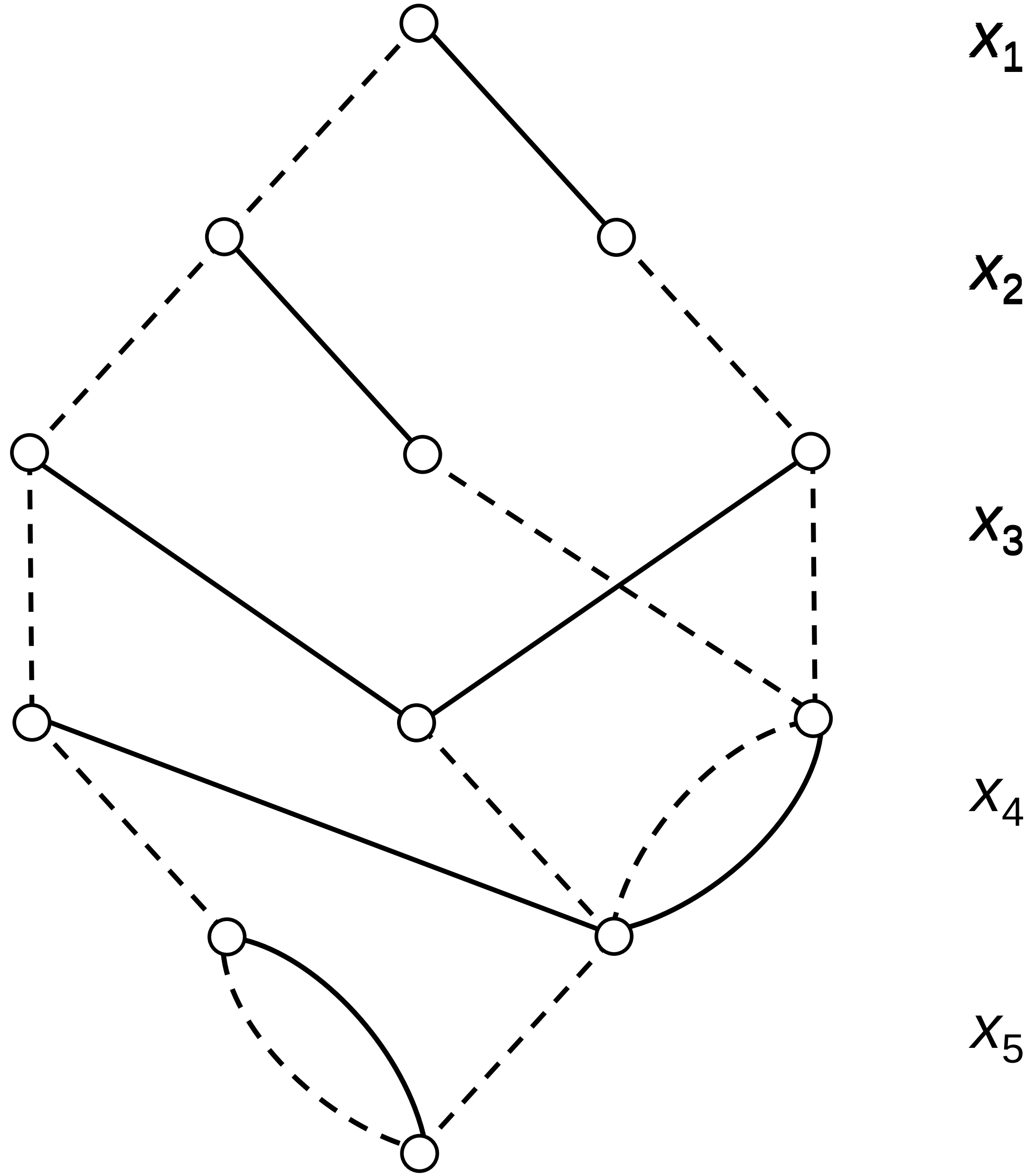
Let each vertex have weight w_i

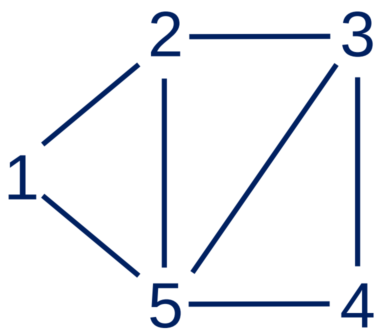
Select nonadjacent vertices to maximize $\sum_i w_i x_i$



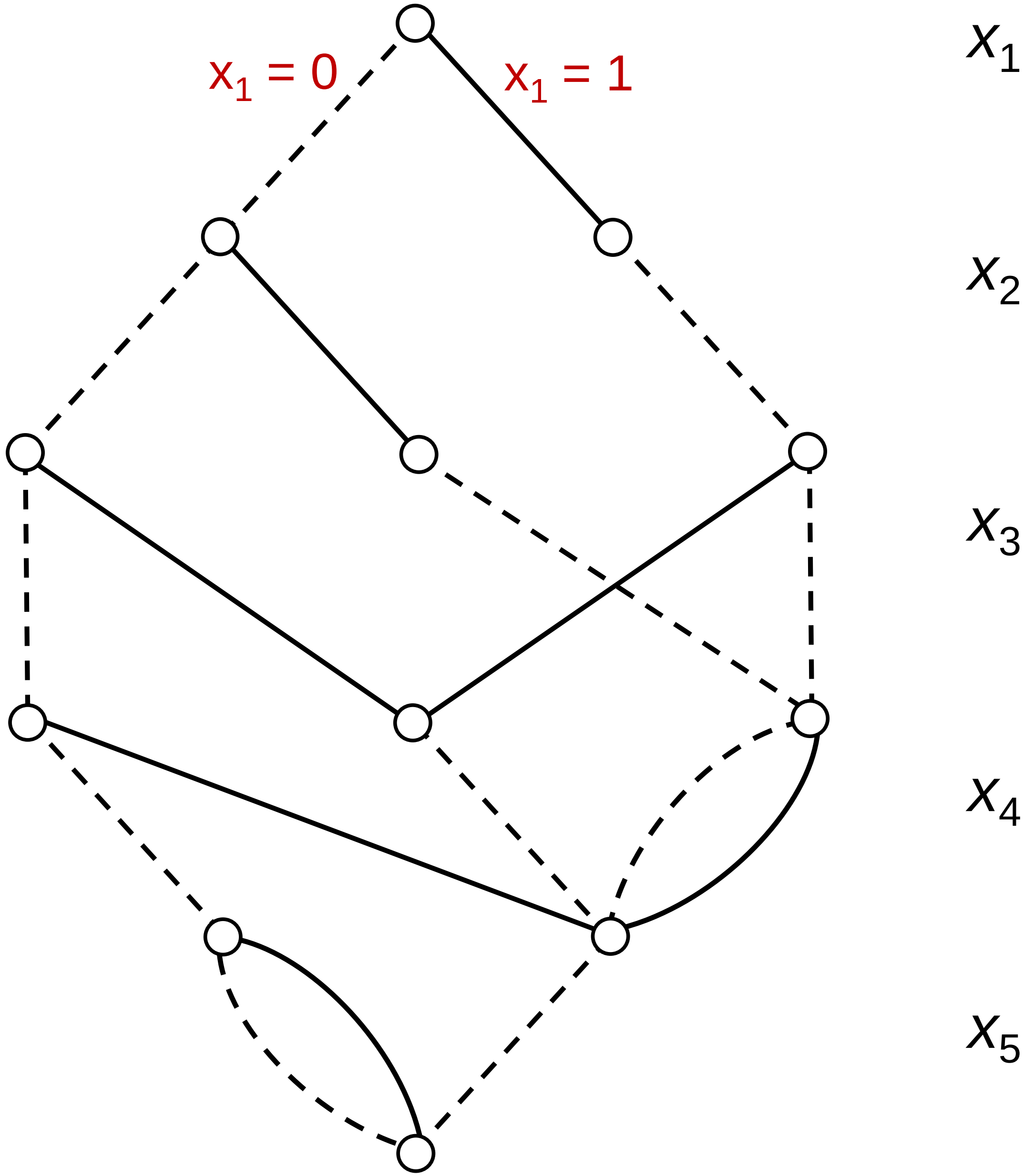


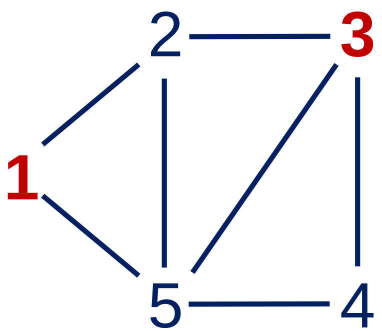
Exact DD for stable set problem



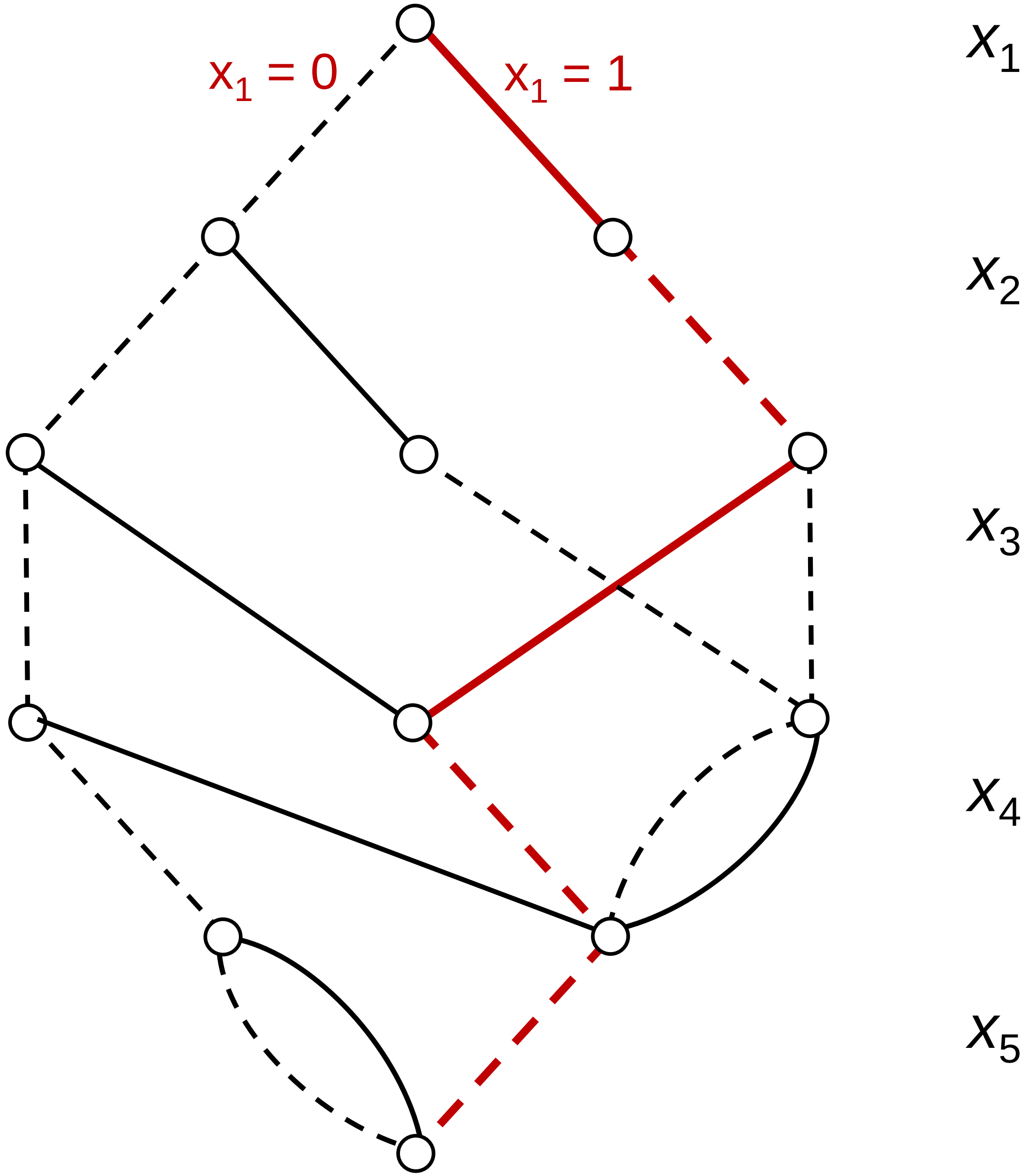


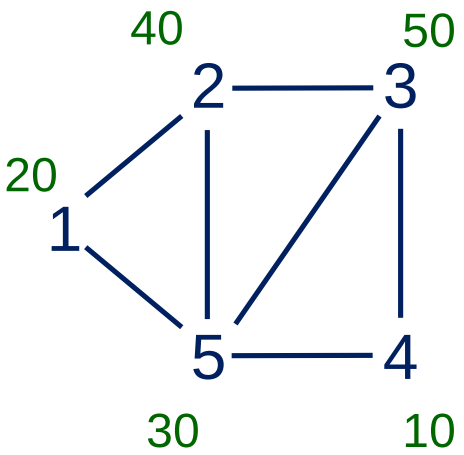
Exact DD for
stable set
problem



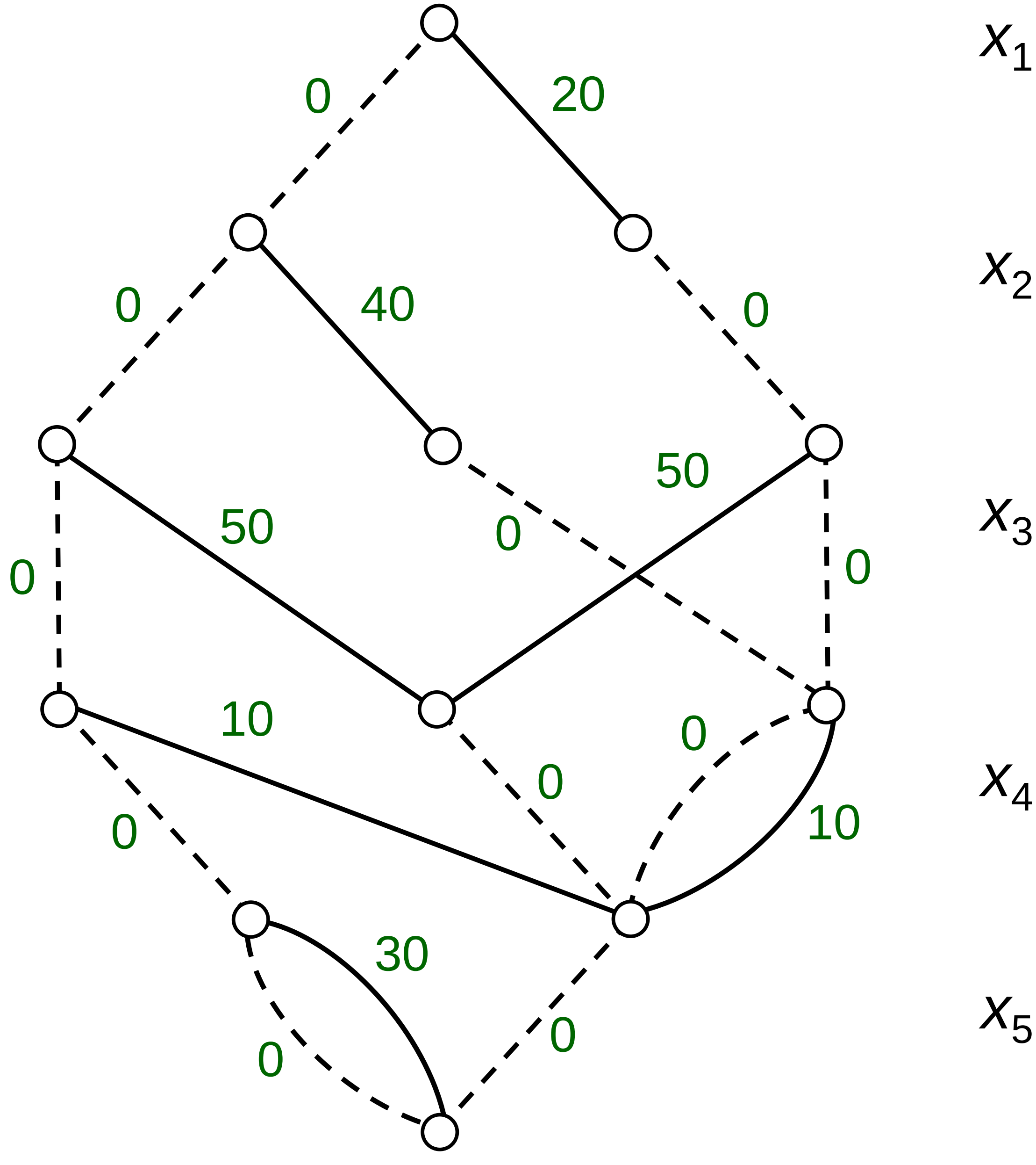


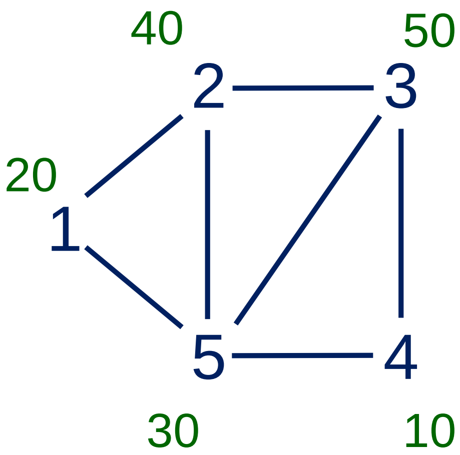
Paths from top to bottom correspond to the 9 feasible solutions

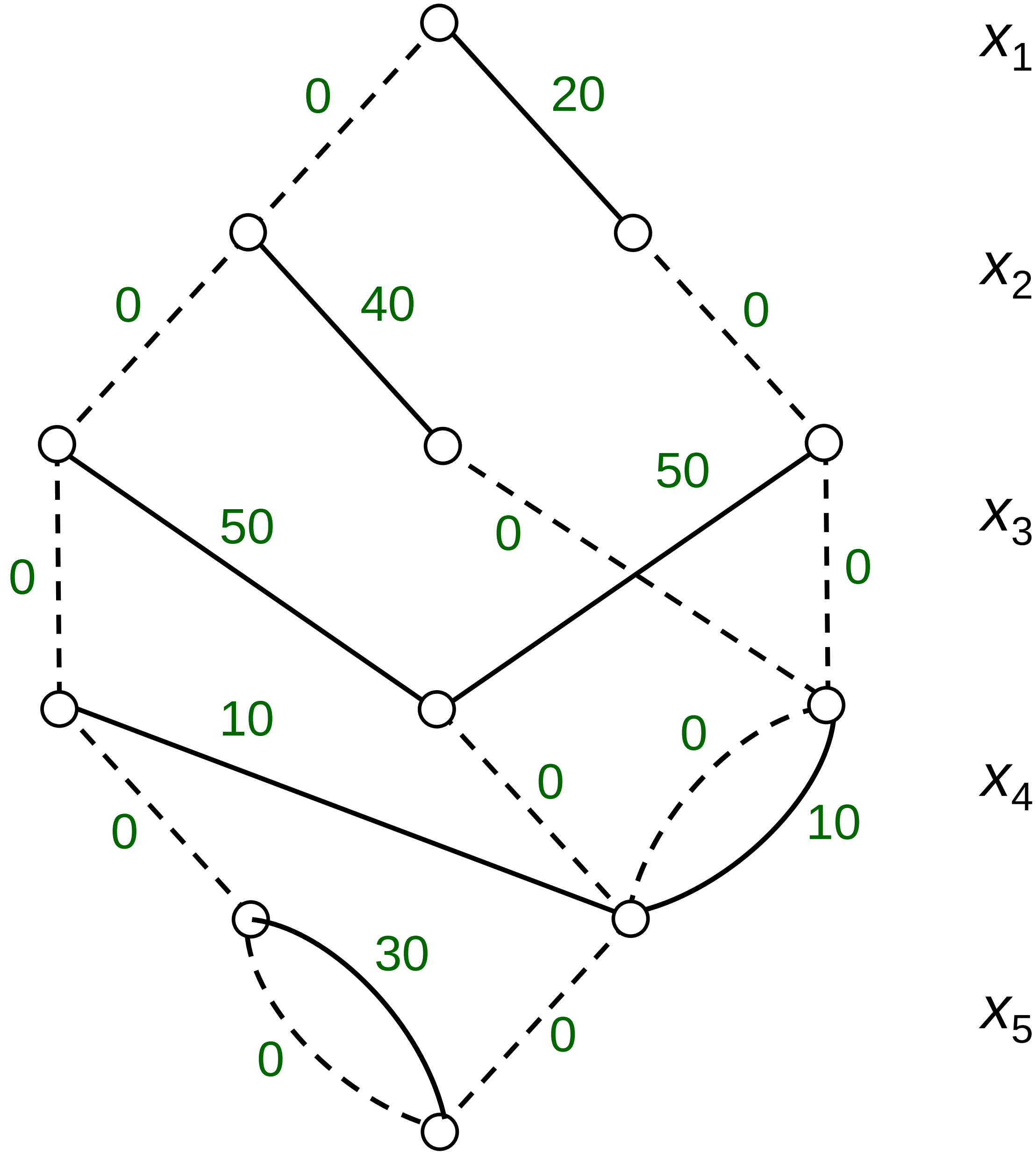




For objective function, associate weights with arcs

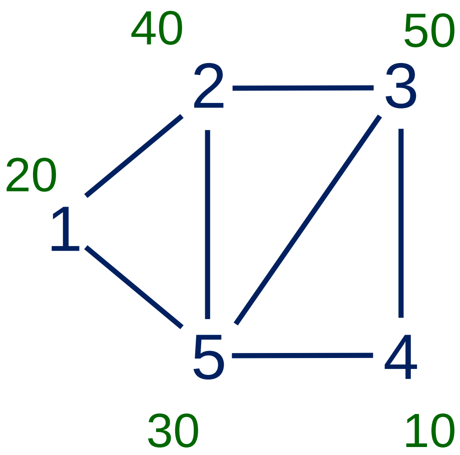


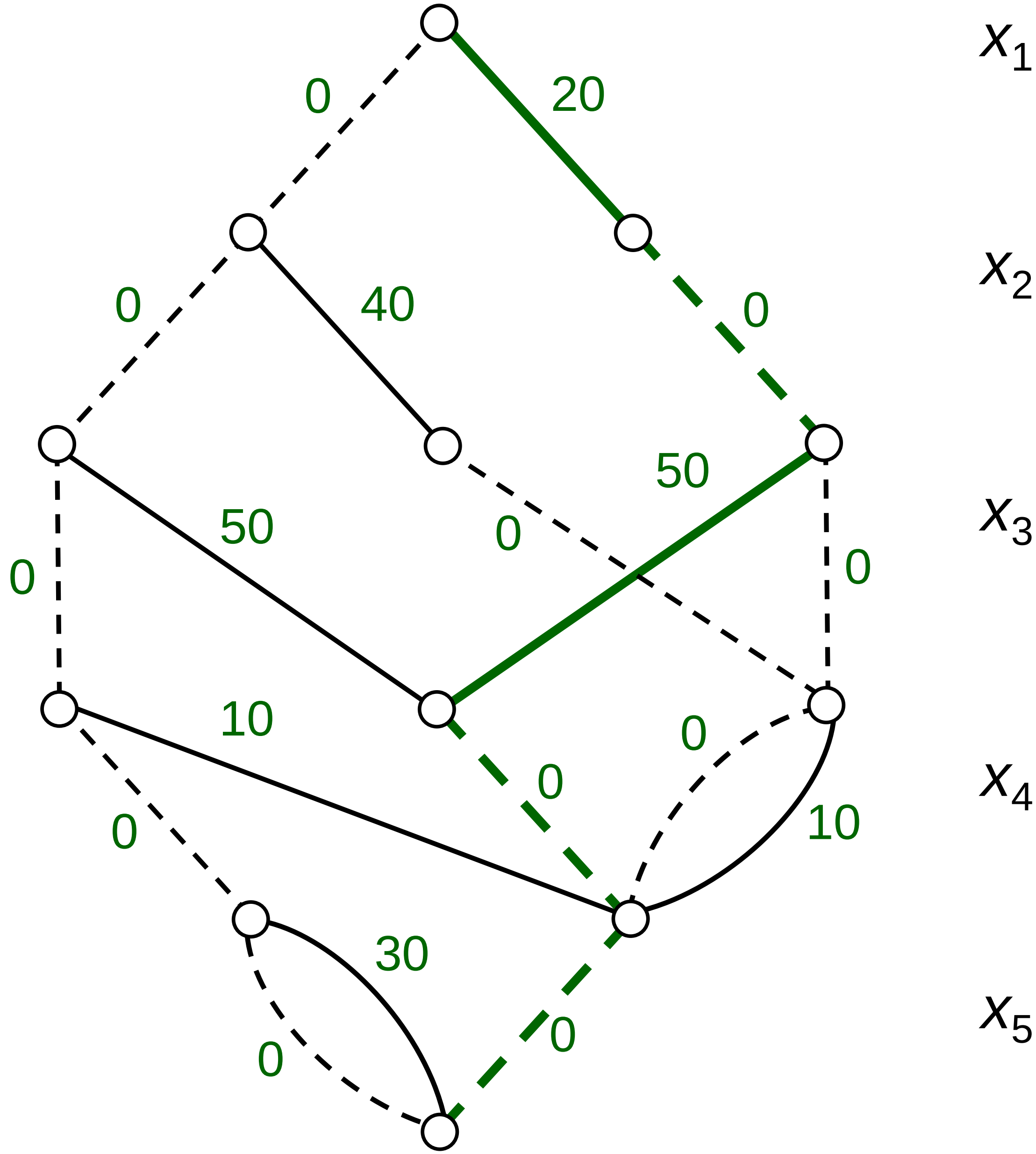




For objective function, associate weights with arcs

Optimal solution is **longest path**



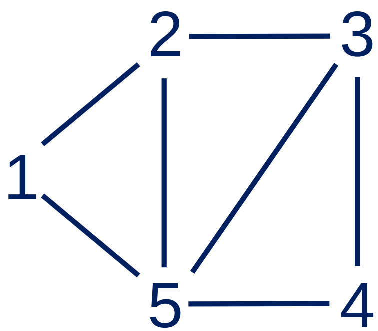


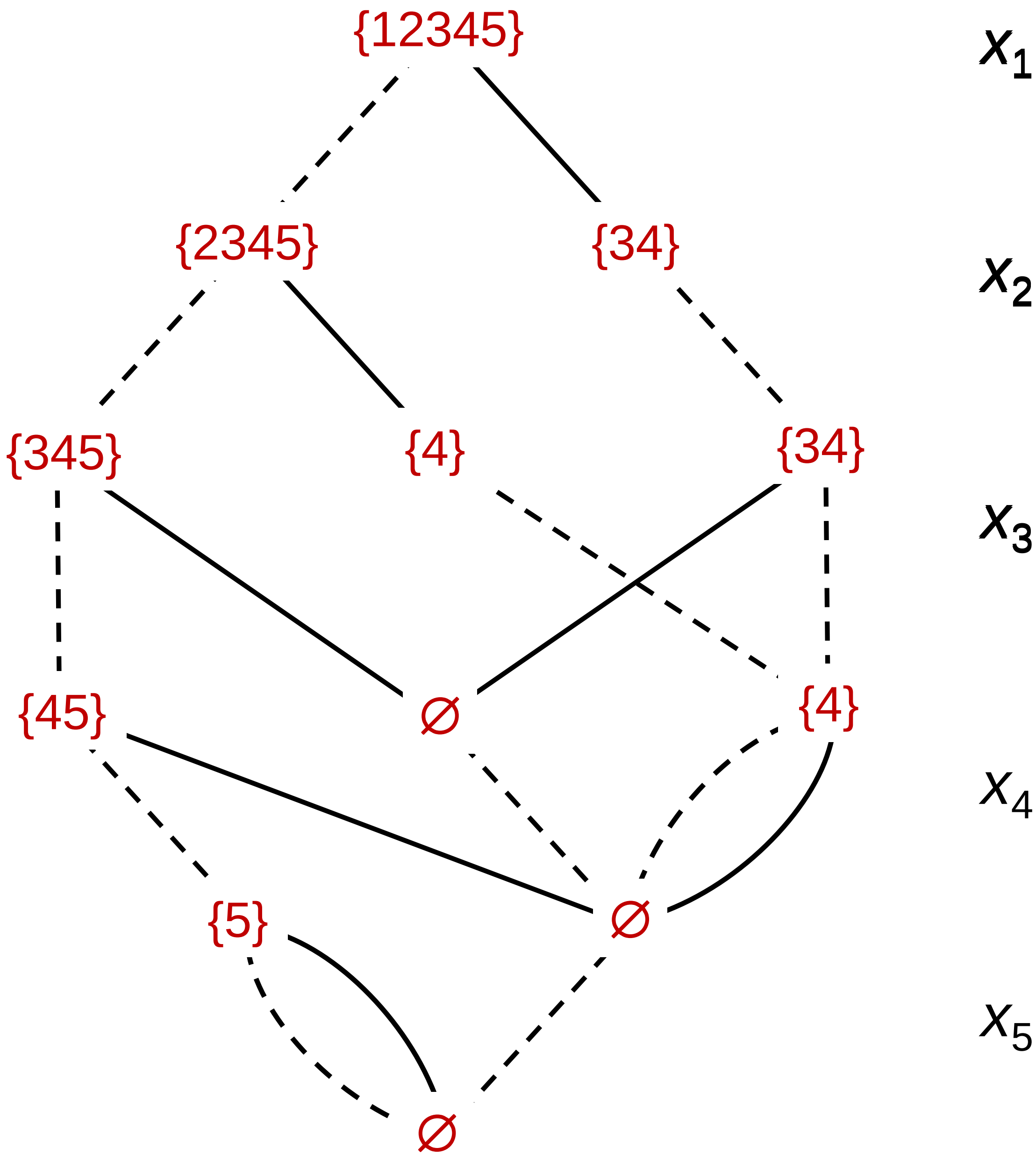
For objective function, associate weights with arcs

Optimal solution is **longest path**

Exact DD Compilation

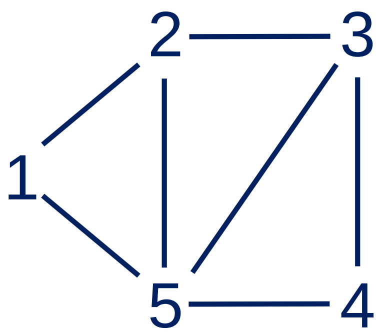
- Build an exact DD by associating a **state** with each node.
  - Merge nodes with **identical states**.





Exact DD for stable set problem

To build DD, associate **state** with each node



{12345}

x_1

x_2

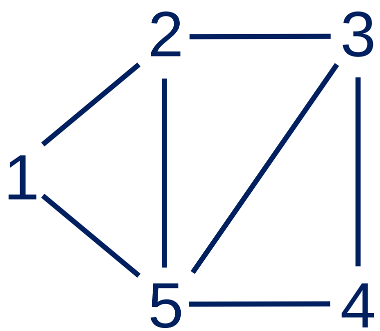
Exact DD for stable set problem

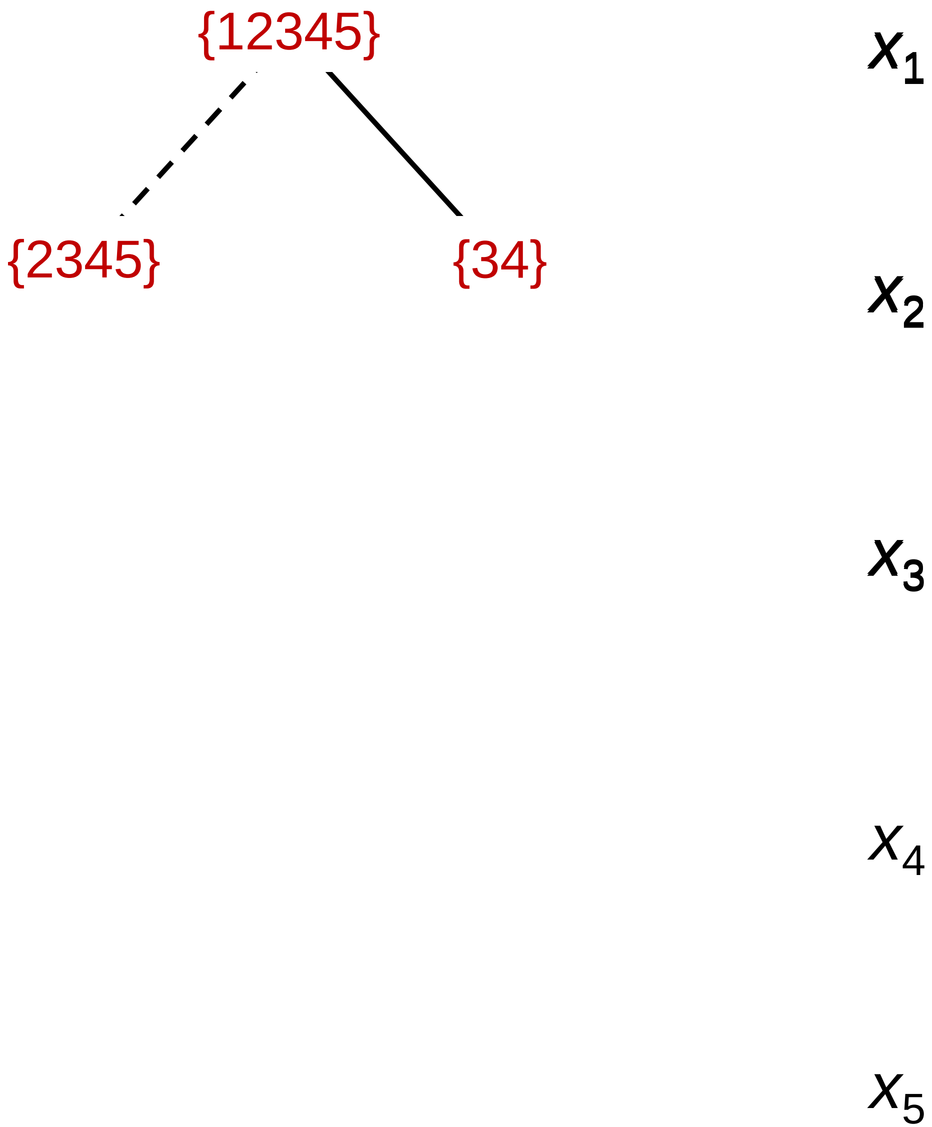
x_3

x_4

To build DD, associate **state** with each node

x_5





Exact DD for stable set problem

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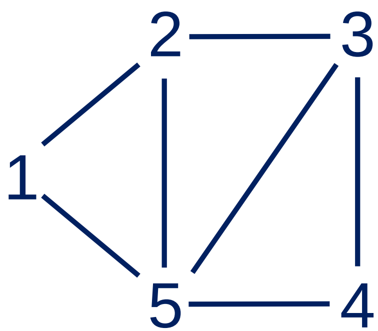
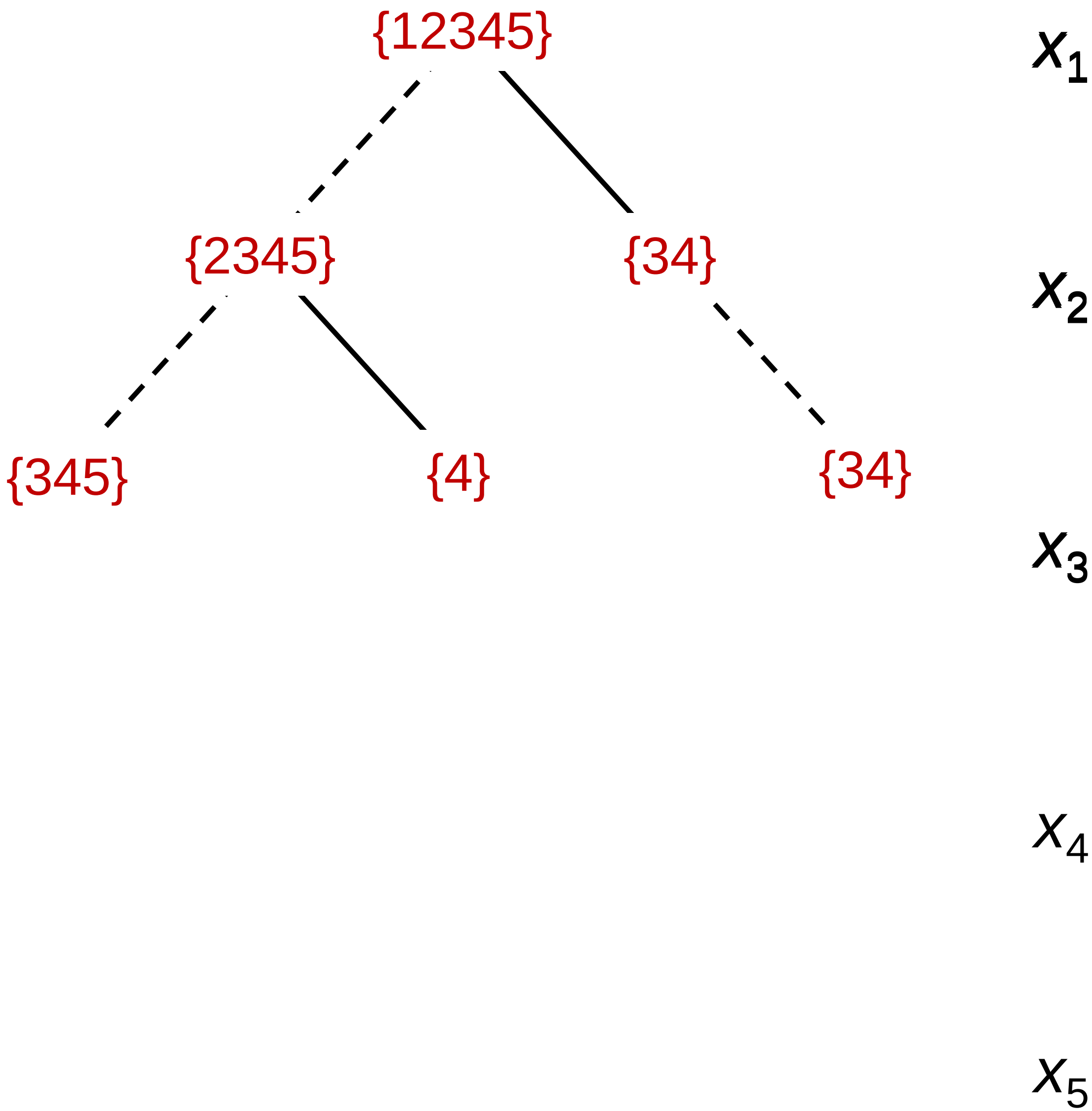
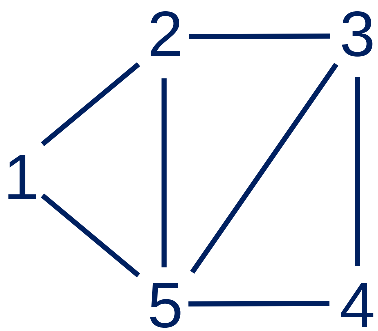
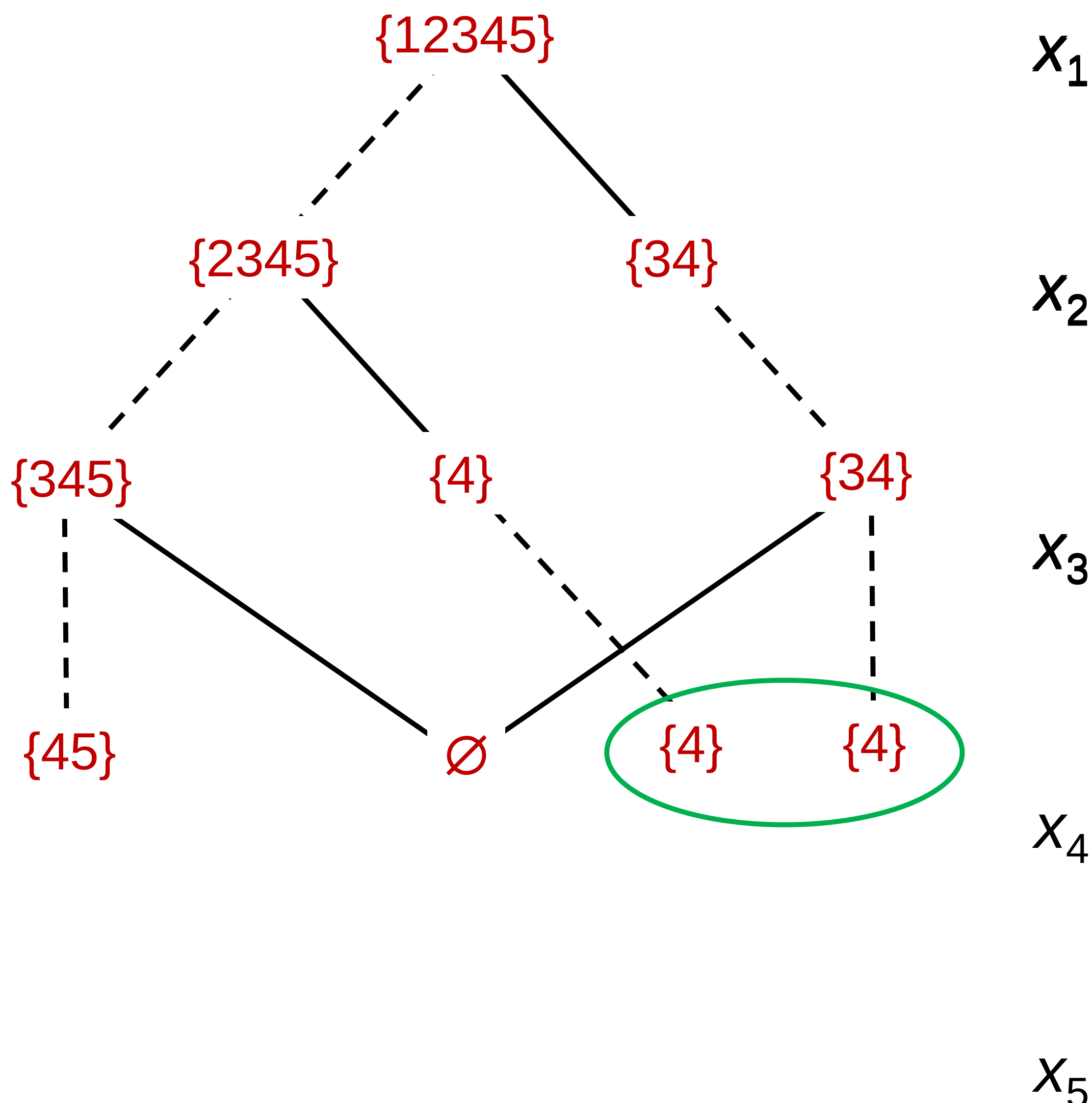




Exact DD for stable set problem

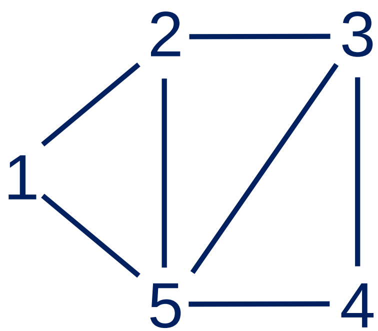
To build DD, associate **state** with each node



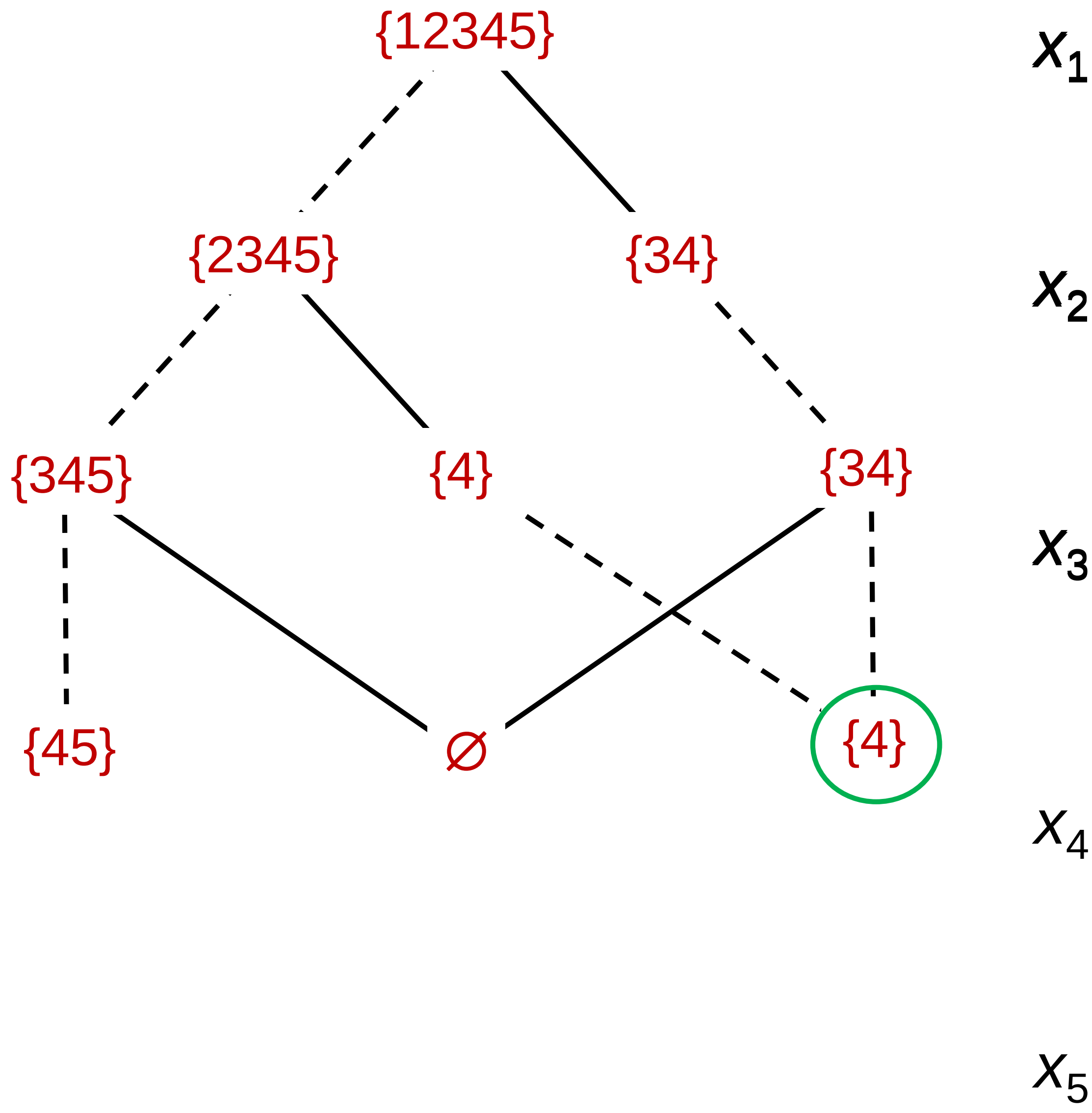


Exact DD for stable set problem

Merge nodes that correspond to the same state



Exact DD for stable set problem



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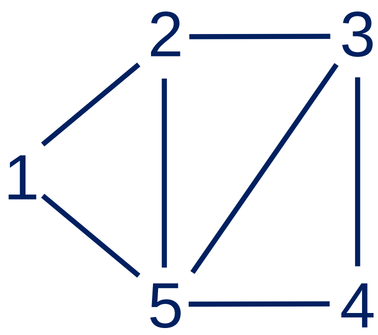
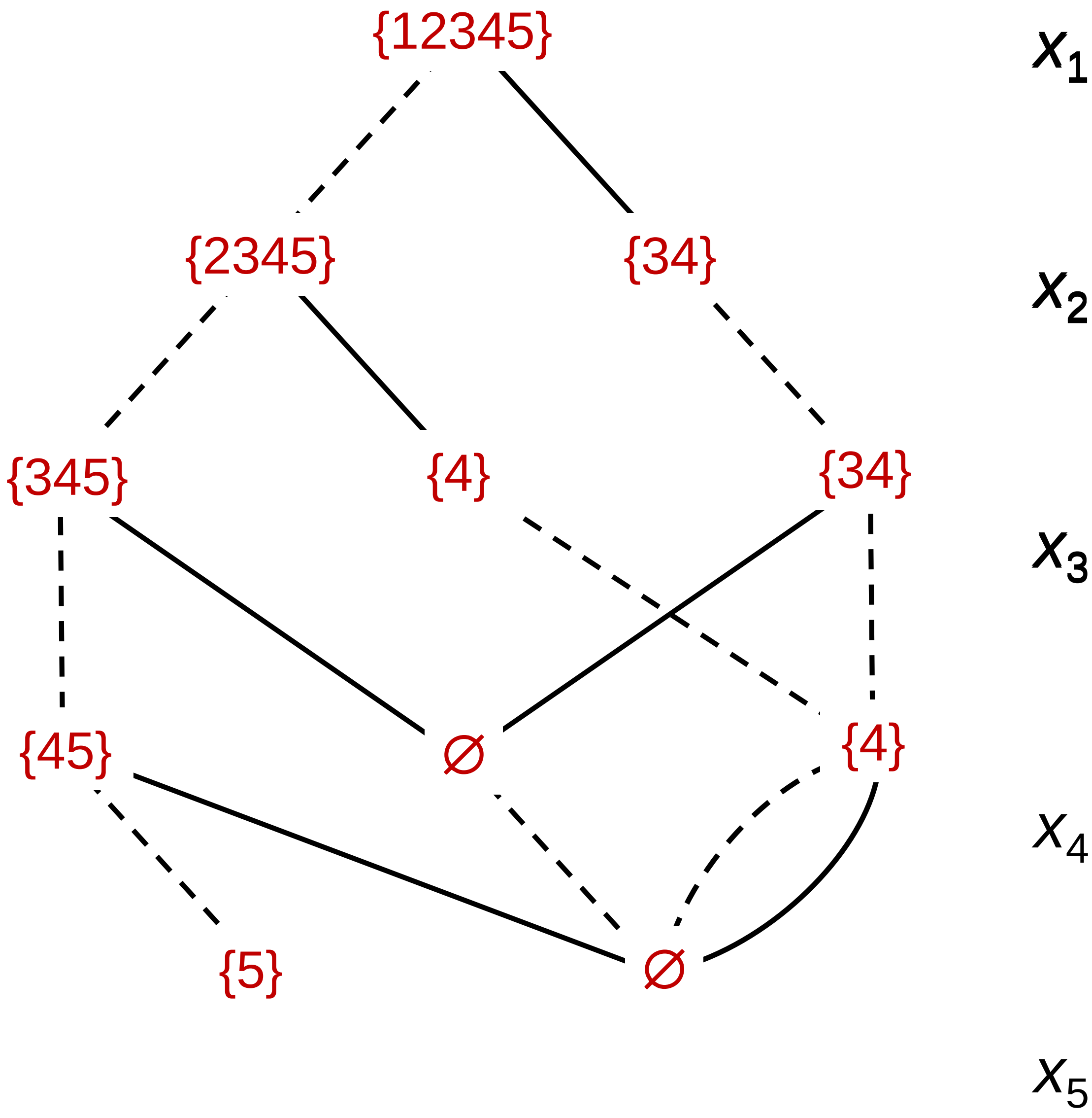
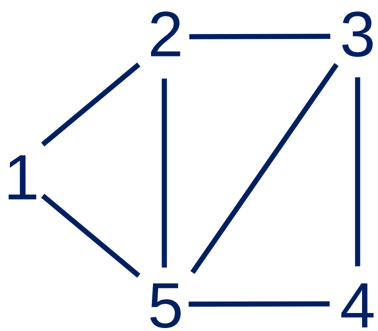


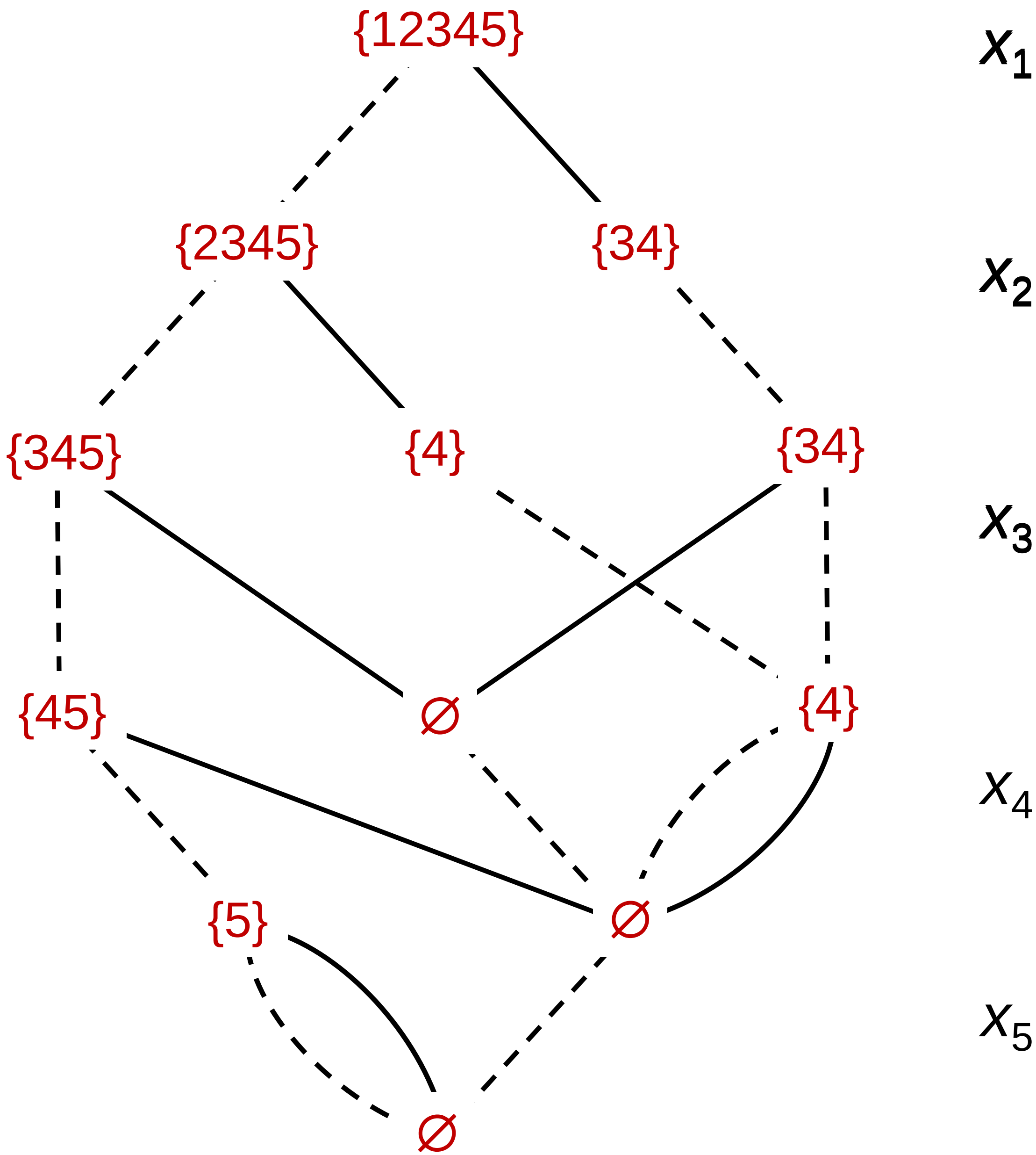
Exact DD for stable set problem

To build DD, associate **state** with each node





Exact DD for stable set problem



Resulting DD is **not necessarily reduced** (it is in this case).

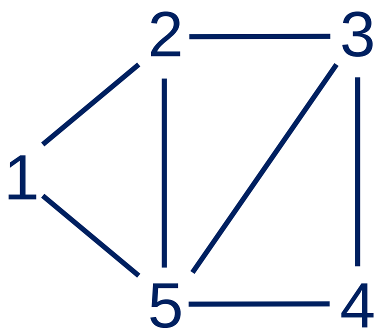
A General-Purpose Solver

- The decision diagram tends to grow exponentially**.**
- To build a **practical solver:**
  - Use limited-width **relaxed** decision diagrams to bound the objective value.
  - Use limited-width **restricted** decision diagrams for primal heuristic
  - Use a **recursive dynamic programming model**.
  - Use **novel branching scheme** within relaxed decision diagrams.

Relaxed Decision Diagrams

- A **relaxed** DD represents a superset of feasible set.
  - Shortest (longest) path length is a **bound** on optimal value.
  - **Size of DD is controlled**.
  - Analogous to LP relaxation in IP, but **discrete**.
  - Does **not** require **linearity**, **convexity**, or **inequality** constraints.

Andersen, Hadžić, JH, Tiedemann (2007)



{12345}

x_1

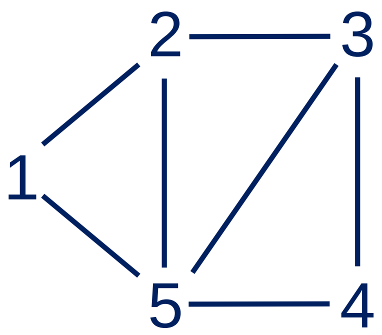
x_2

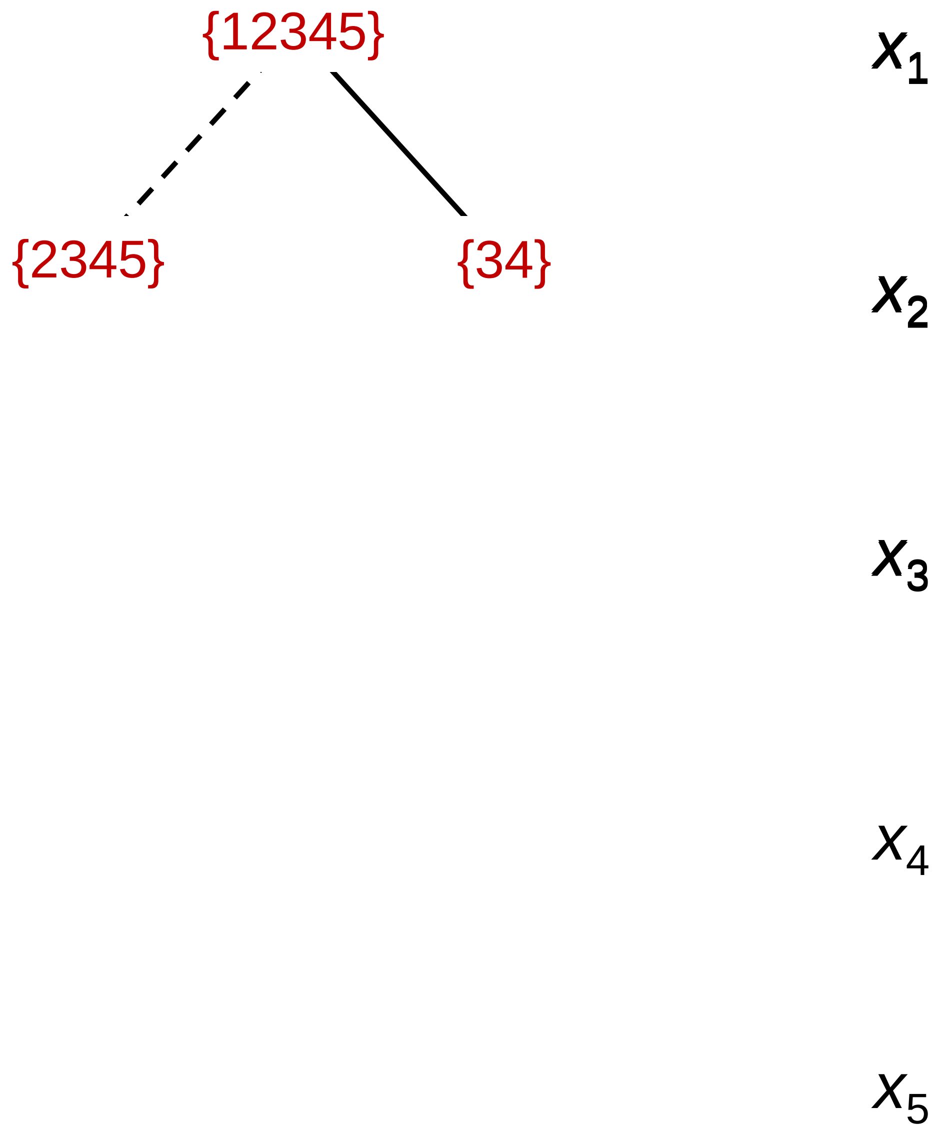
x_3

x_4

x_5

To build **relaxed** DD, merge some additional nodes as we go along





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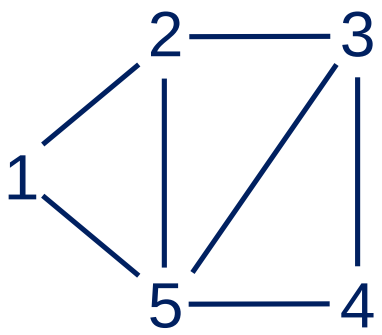
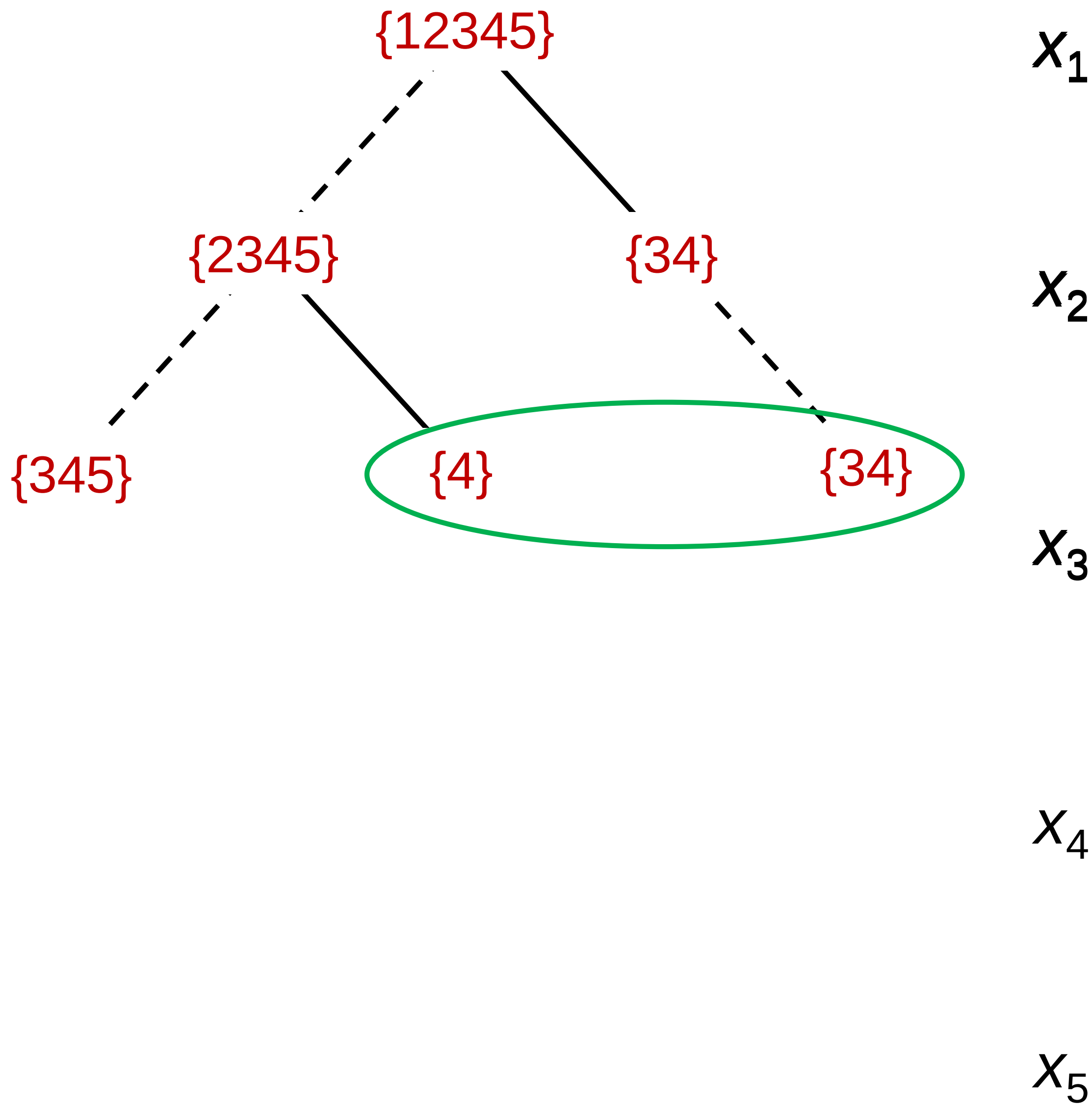
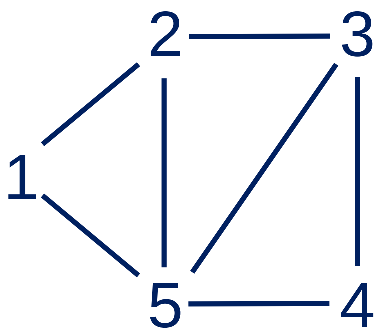
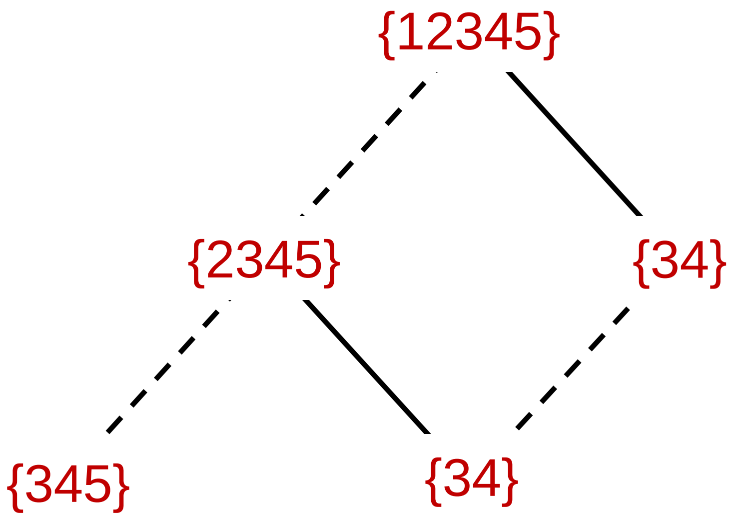




To build **relaxed** DD, merge some additional nodes as we go along.

Take the **union** of merged states





x_1

x_2

x_3

x_4

x_5

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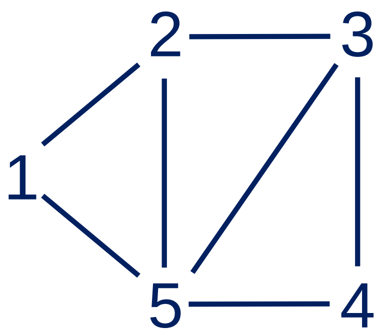

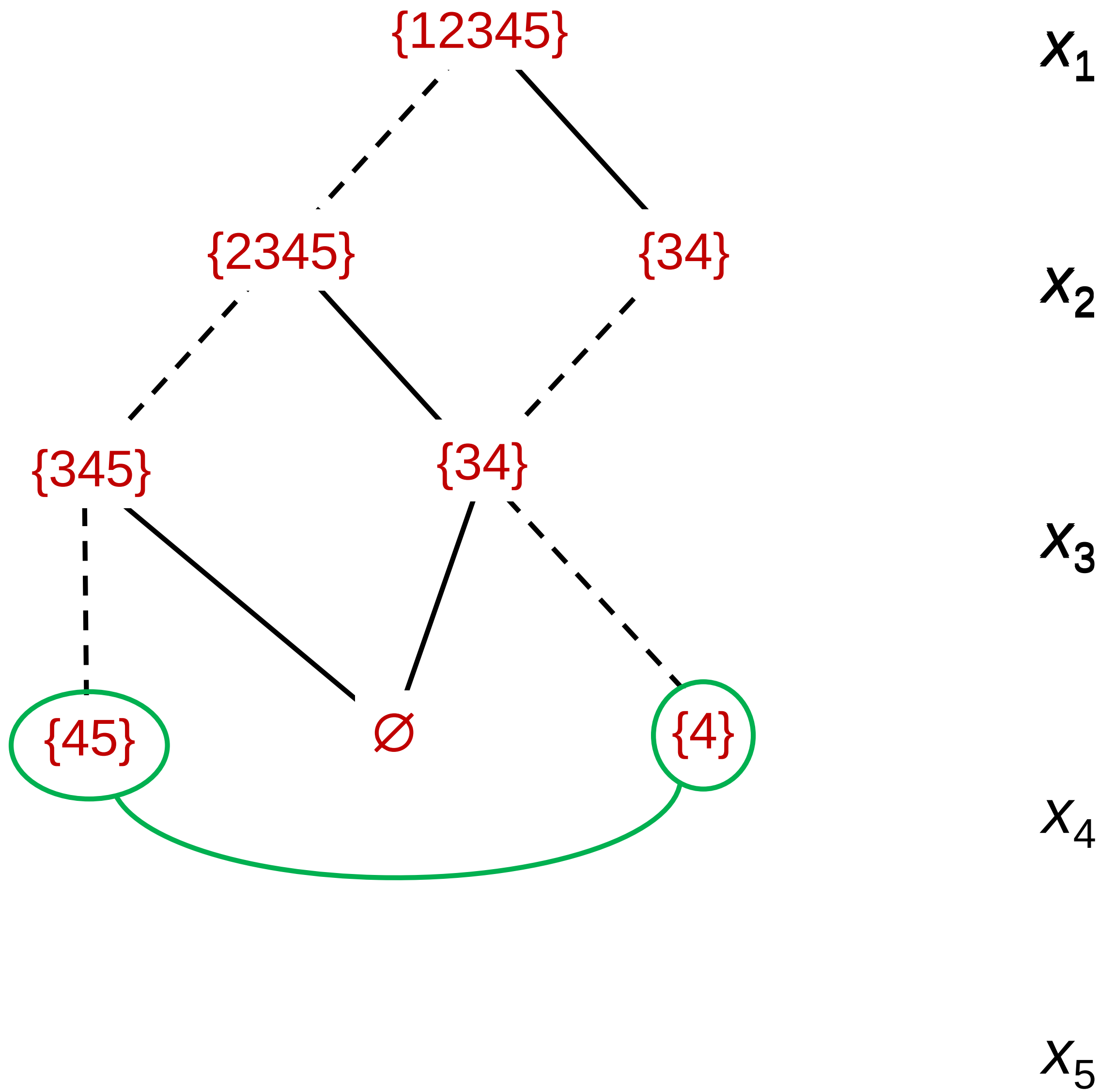

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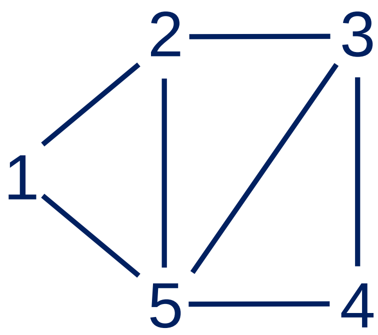
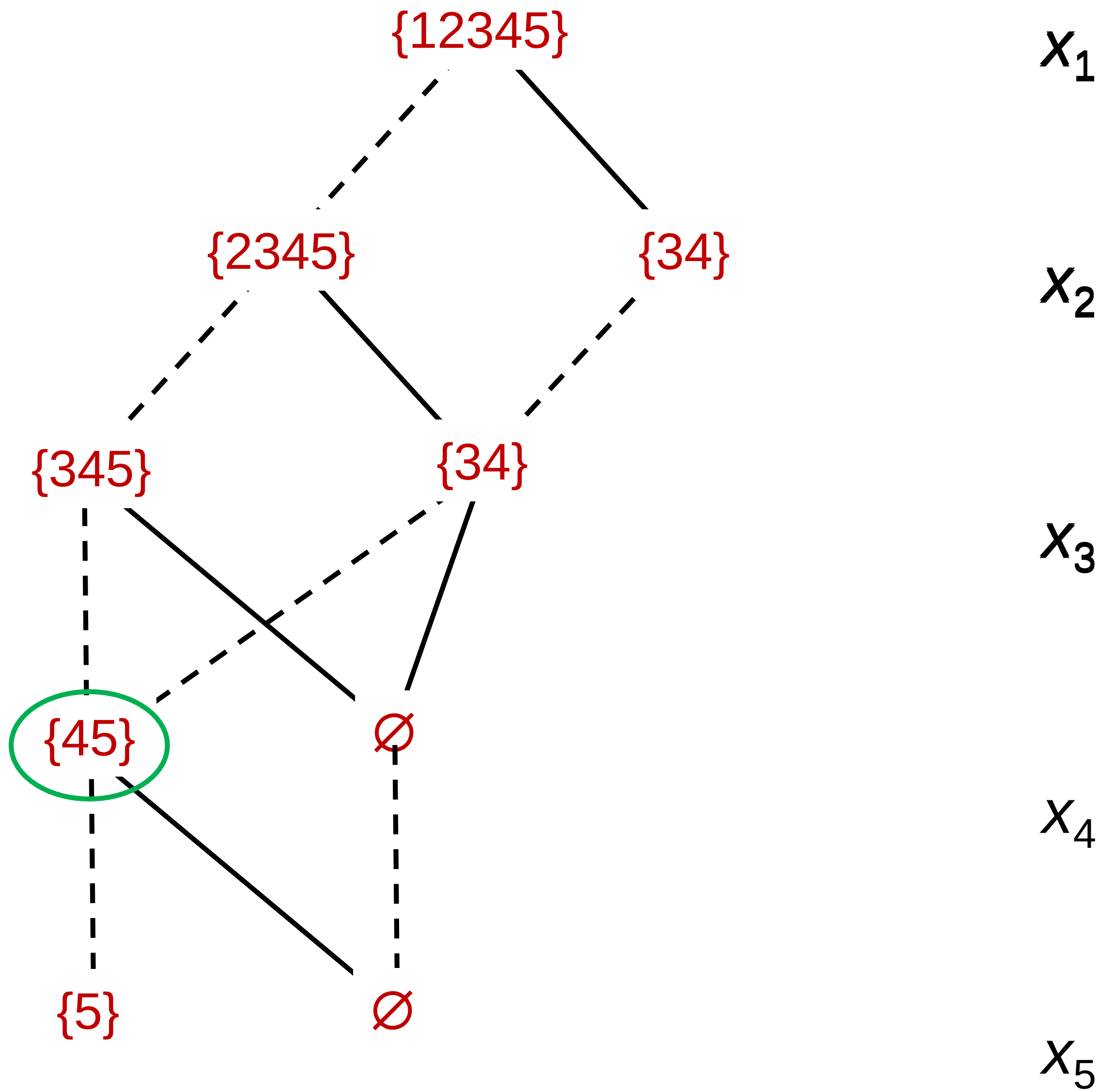
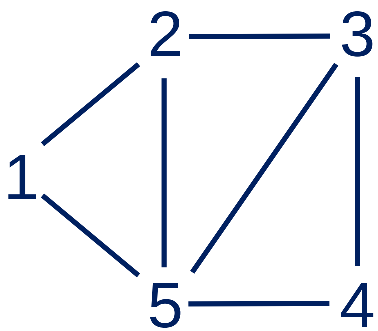




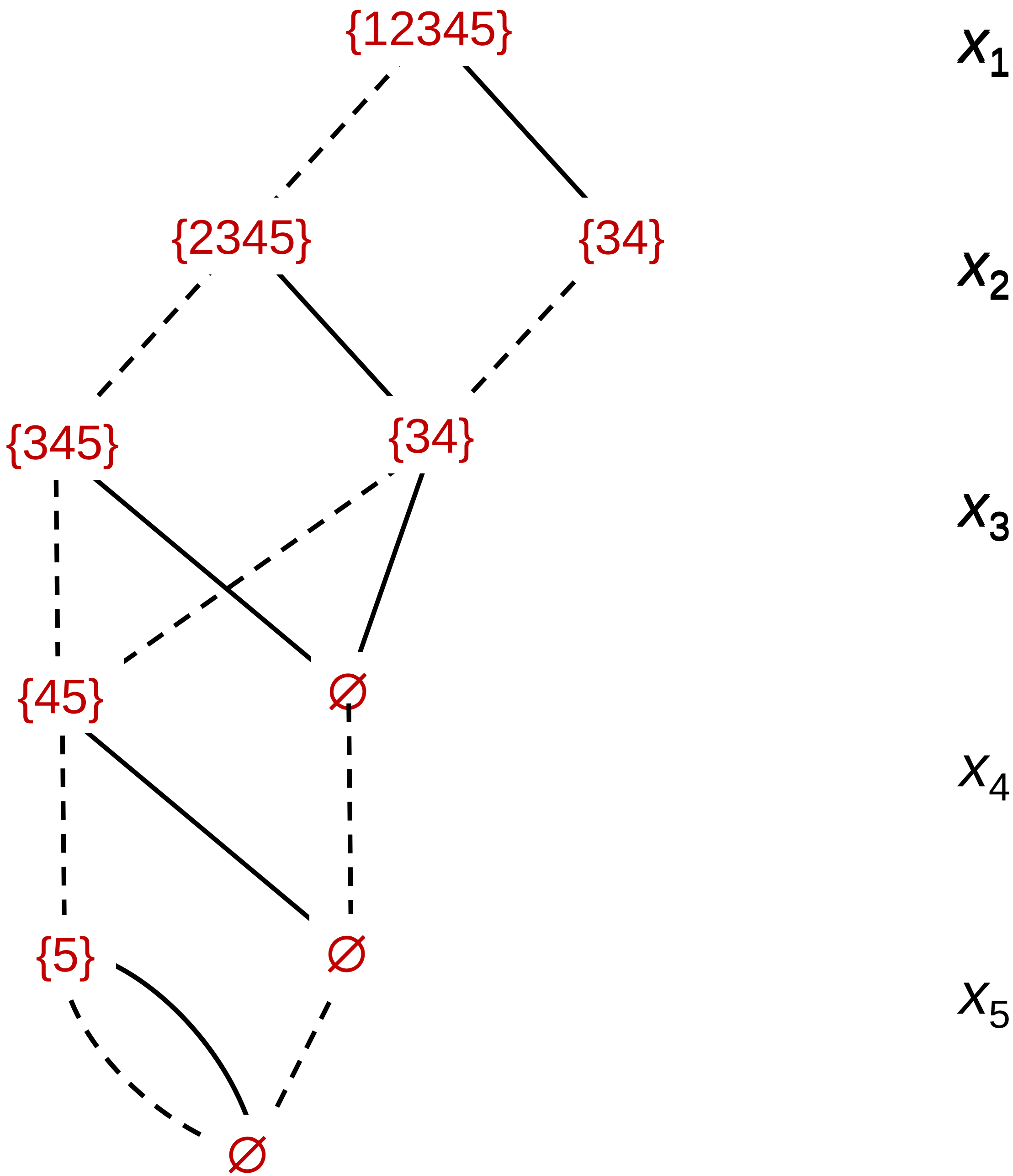
To build **relaxed** DD, merge some additional nodes as we go along.

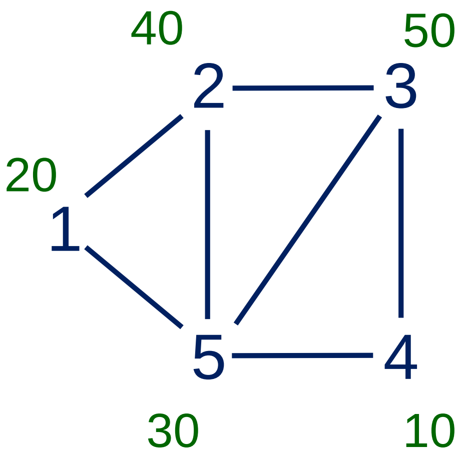
Take the **union** of merged states.



Width = 2

Represents 11 solutions, including 9 feasible solutions





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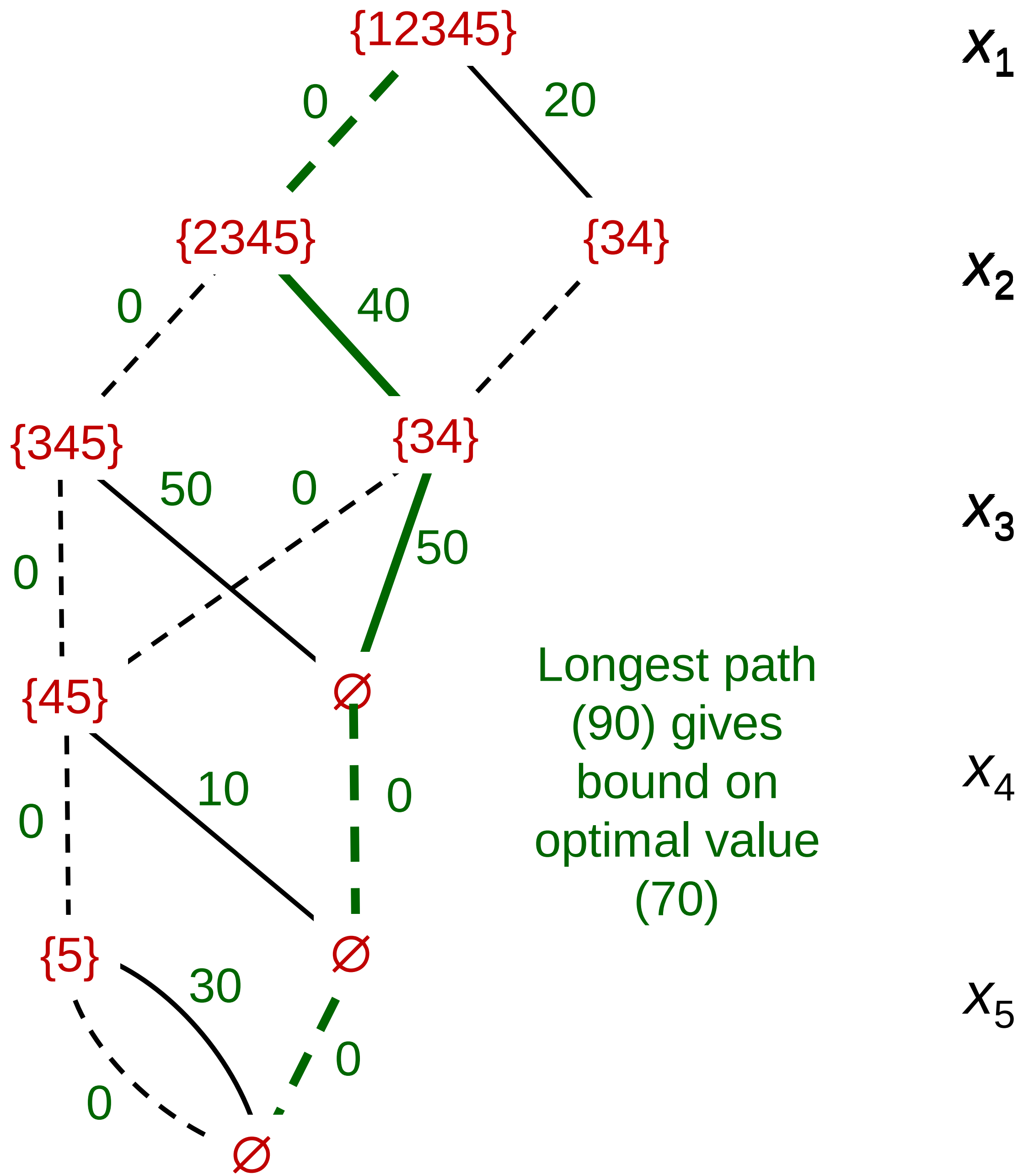


Longest path
(90) gives
bound on
optimal value
(70)

Relaxed Decision Diagrams

- Alternate relaxation method: **node refinement**.
  - Start with DD of width 1 representing Cartesian product of variable domains.
  - Split nodes so as to remove some infeasible paths.
  - Will be illustrated in **next tutorial**.

Andersen, Hadžić, JH, Tiedemann (2007)

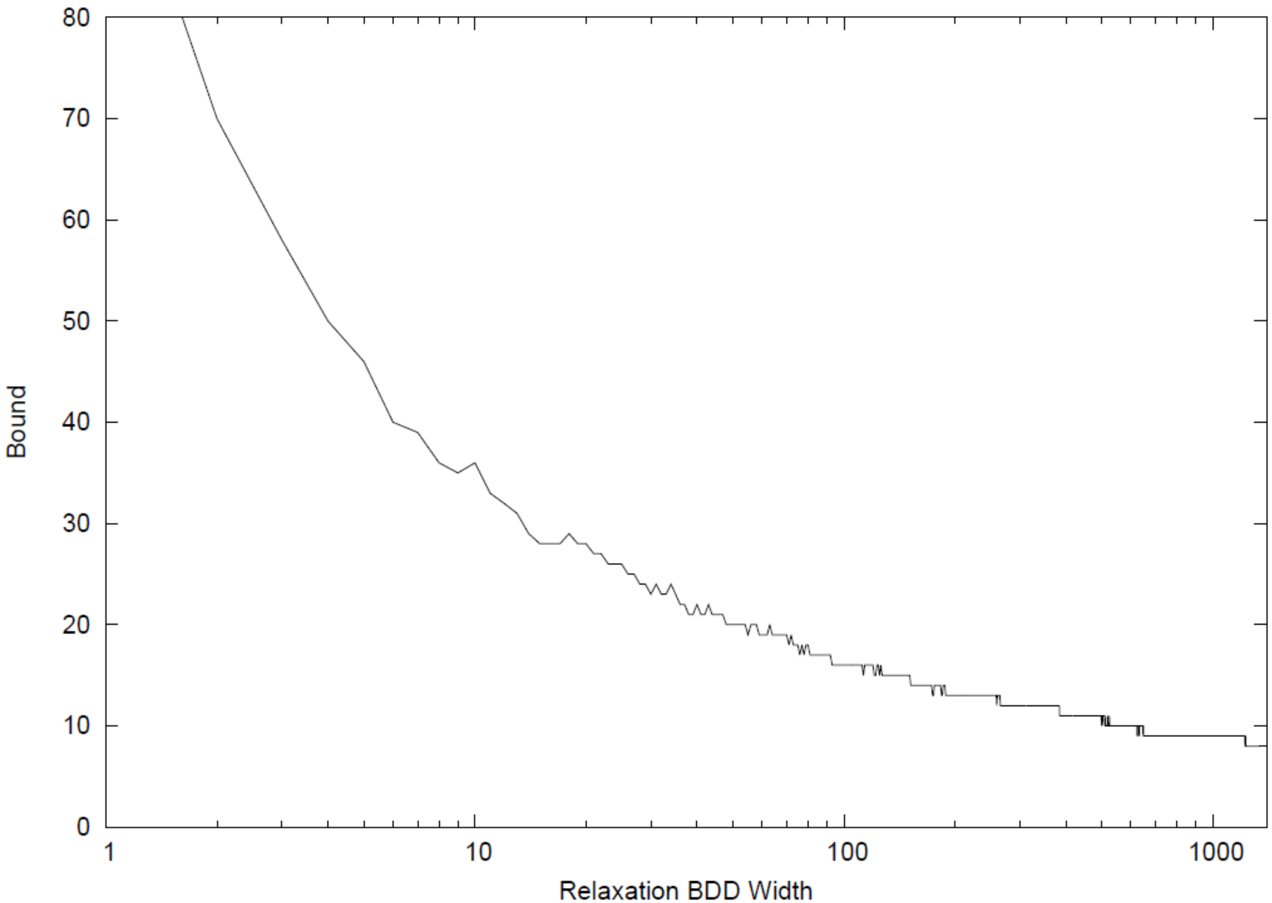
Relaxed Decision Diagrams

- Original application: enhanced propagation in constraint programming
  - In multiple alldiff problem (graph coloring), reduced 1 million node search trees to 1 node.

Andersen, Hadžić, JH, Tiedemann (2007)

Relaxed Decision Diagrams

- Wider diagrams yield tighter bounds
  - But take longer to build.
  - Adjust width dynamically.

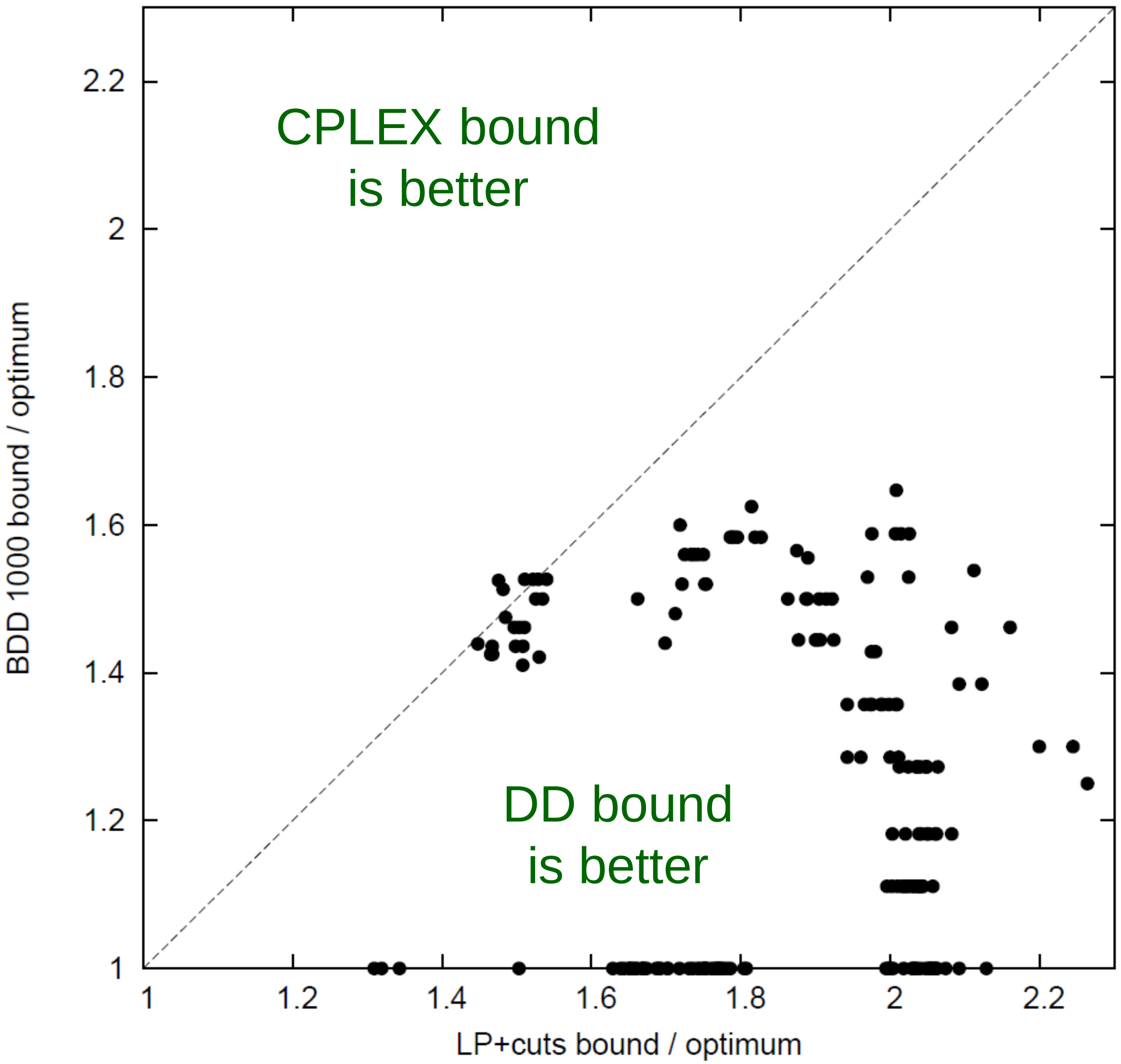


Bergman, Ciré, van Hoeve, JH (2013)

Relaxed Decision Diagrams

- DDs vs. CPLEX bound at root node for max stable set problem
  - Using CPLEX default cut generation
  - DD max width of 1000.
  - DDs require about 5% the time of CPLEX

Bergman, Ciré, van Hoeve, JH (2013)



Restricted Decision Diagrams

- A **restricted** DD represents a **subset** of the feasible set.
- Restricted DDs provide a basis for a **primal heuristic**.
  - Shortest (longest) paths in the restricted DD provide good feasible solutions.
  - Generate a **limited-width** restricted DD by deleting nodes that appear unpromising.

Bergman, Ciré, van Hoeve, Yunes (2014)

Set covering problem

$$x_1 + x_2 + x_3 \geq 1$$
$$x_1 + x_4 + x_5 \geq 1$$
$$x_2 + x_4 + x_6 \geq 1$$

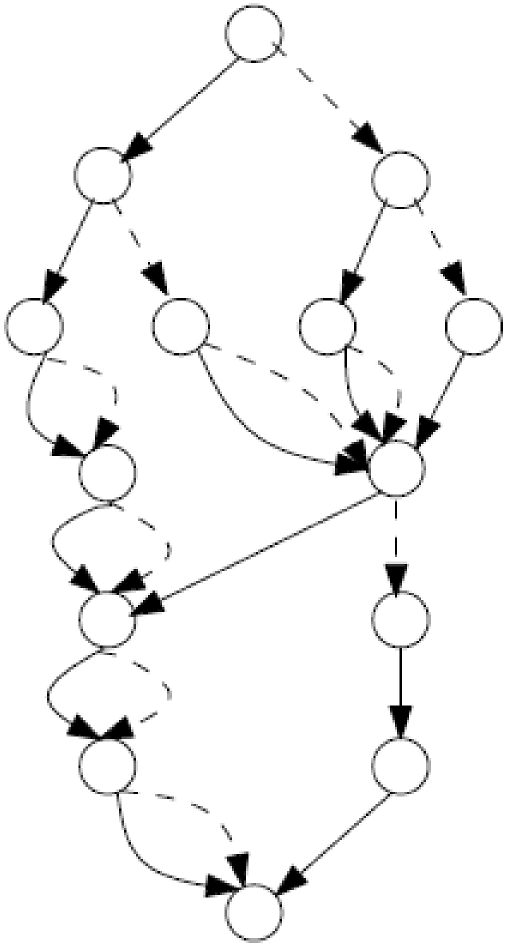
Sets

	1	2	3	4	5	6
A	●	●	●			
B	●			●	●	
C		●		●		●

52 feasible solutions.

Minimum cover of 2, e.g. x_1, x_2

Restricted DD of width 4

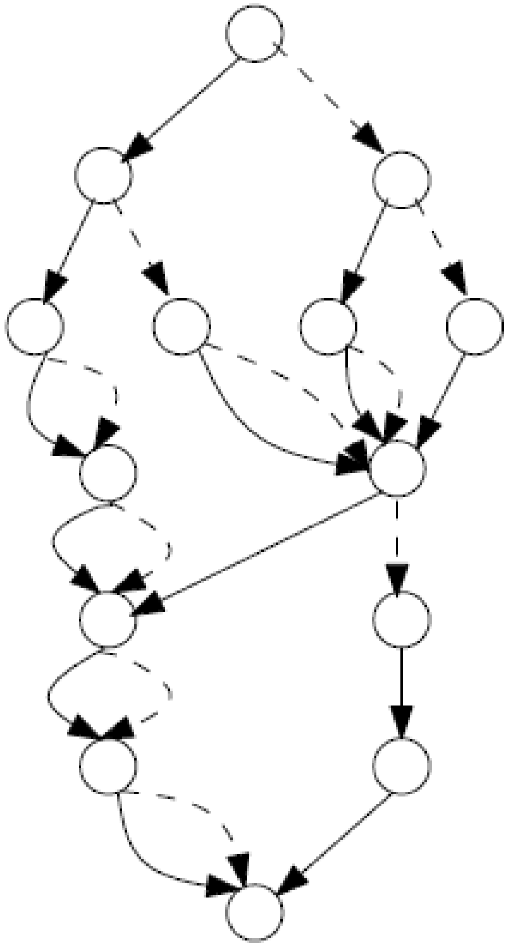


Several shortest paths have length 2.

All are minimum covers.

41 paths (< 52 feasible solutions)

Restricted DD of width 4



Several shortest paths have length 2.

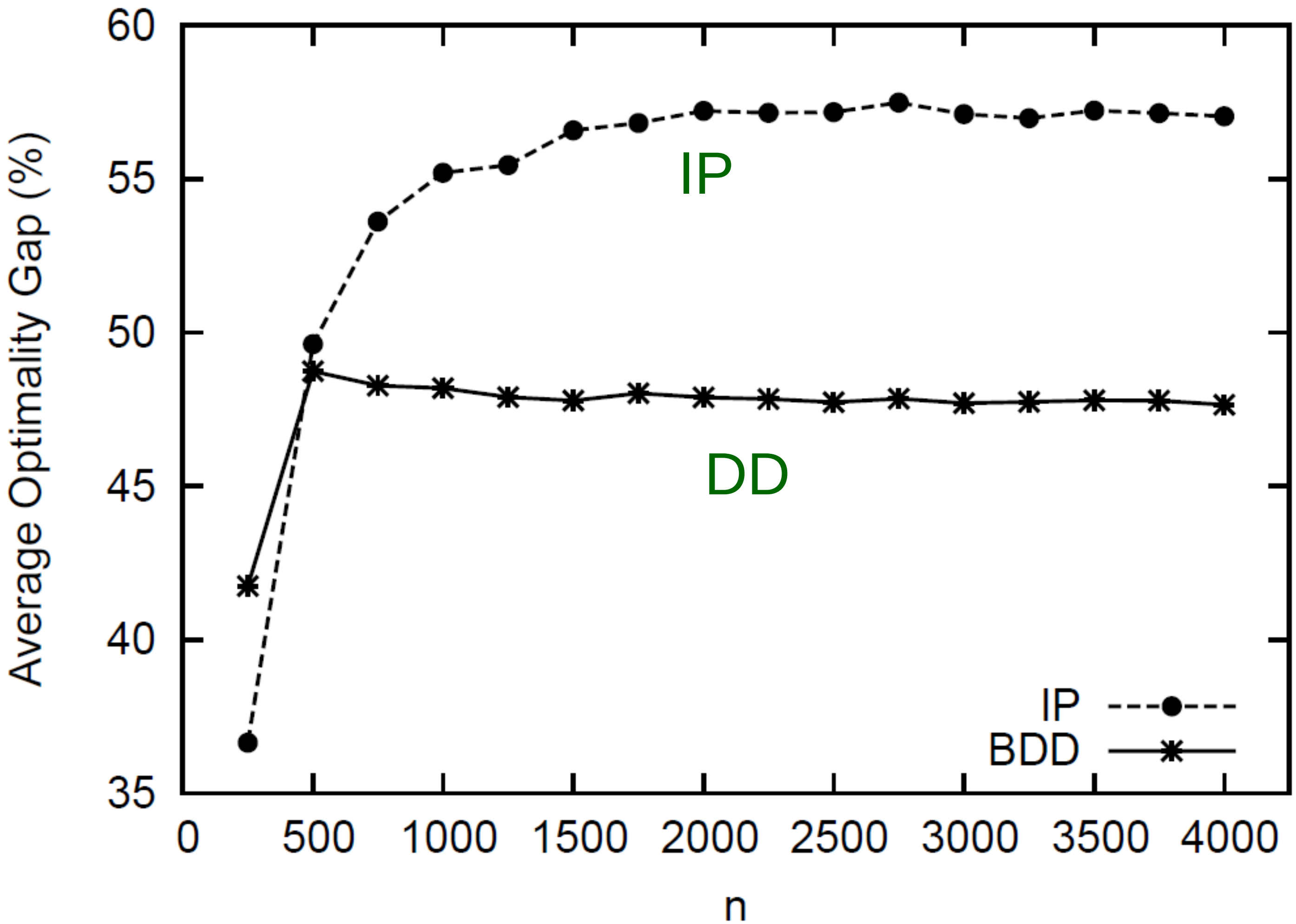
All are minimum covers.

In this case, restricted DD delivers optimal solutions.

41 paths (< 52 feasible solutions)

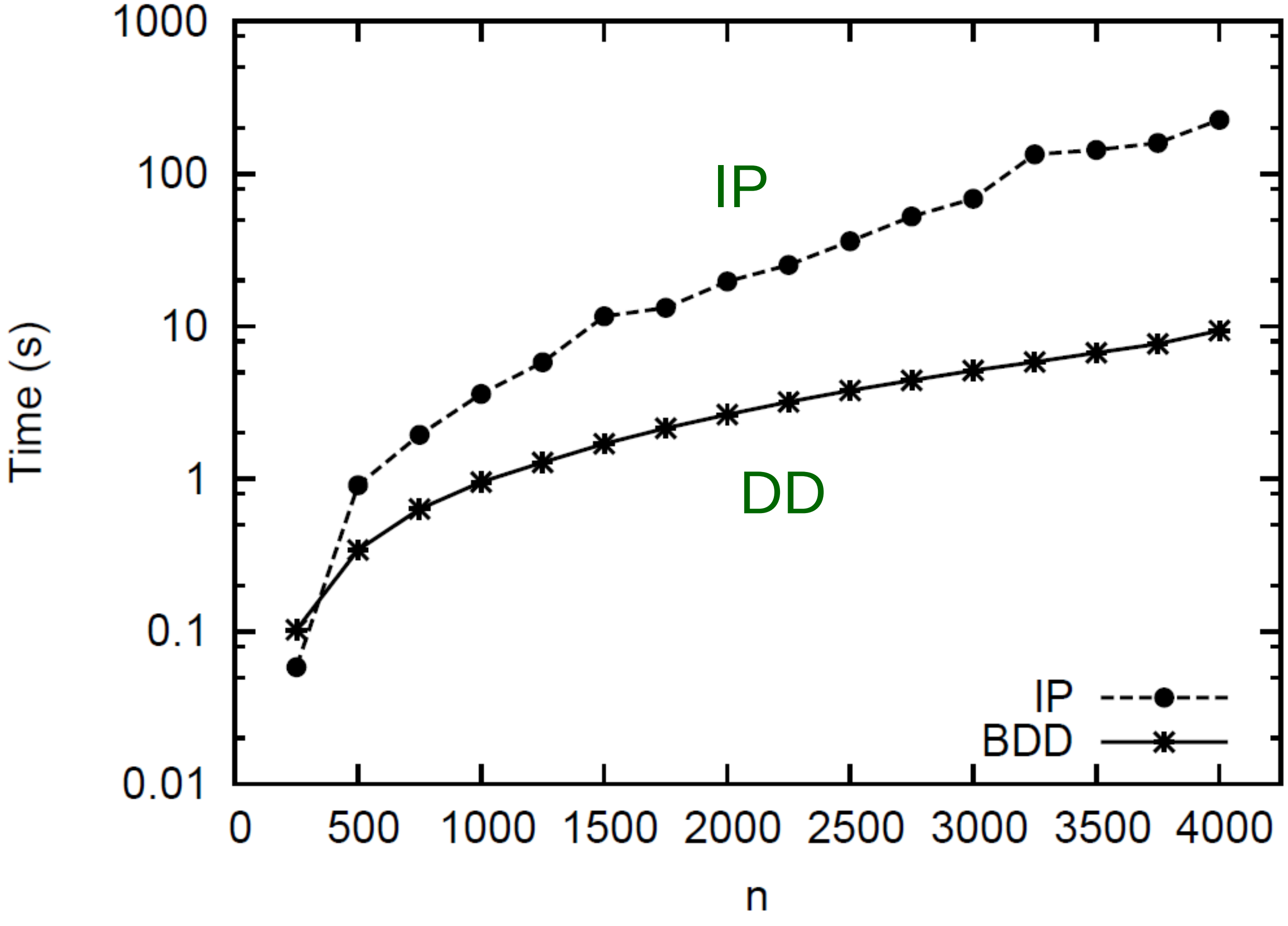
Optimality gap for set covering, *n* variables

Restricted DDs vs
Primal heuristic at root node of CPLEX



Computation time

Restricted DDs vs
Primal heuristic at root node of CPLEX (cuts turned off)



Dynamic Programming Model

- Formulate problem with **dynamic programming** model.
  - Rather than constraint set.
  - Problem must have **recursive** structure
  - But there is great **flexibility** to represent constraints and objective function.
  - Any function of **current state** is permissible.
  - We **don't care** if state space is **exponential**, because we don't solve the problem by dynamic programming.

Dynamic Programming Model

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  - But there is great **flexibility** to represent constraints and objective function.
  - Any function of **current state** is permissible.
  - We **don't care** if state space is **exponential**, because we don't solve the problem by dynamic programming.
- State variables are the same as in relaxed DD.
  - Must also specify **state merger** rule.

Dynamic Programming Model

- Max stable set problem on a graph.
  - **State** = set of vertices that can be added to stable set.

Recursion:

$$g(J) = \max_{j \in J} \left\{ w_j + g\big(J \setminus N(j)\big) \right\}$$

Cost-to-go

State

Immediate cost (edge weight)

Vertex *j* and neighbors

Boundary condition:

$$g(\emptyset) = 0$$

Optimal value:

$$g(\{1, \ldots, n\})$$

Dynamic Programming Model

- Max stable set problem on a graph.
  - **State** = set of vertices that can be added to stable set.
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Cost-to-go

State

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$$g(\emptyset) = 0$$

Merger of states in *M* = $\bigcup_{J \in M} J$

Optimal value:

$$g(\{1, \ldots, n\})$$

Dynamic Programming Model

- Single-machine scheduling with due dates
  - Minimize total tardiness.
  - **State** = (set of jobs not yet processed,
    latest finish time of jobs processed so far)

$$g_i(J_i, f_i) = \max_{j \in J_i} \left\{ (f_i + p_j - d_j)^+ + g_{i+1}(J_i \setminus \{j\}, f_i + p_j) \right\}$$

Cost-to-go

Jobs remaining

Last finish time

Tardiness of job *j*

Boundary condition:

$$g_{n+1}(\emptyset, f_{n+1}) = 0$$

Optimal value:

$$g_1(\{1, \ldots, n\}, 0)$$

Dynamic Programming Model

- Single-machine scheduling with due dates
  - Minimize total tardiness.
  - **State** = (set of jobs not yet processed,
    latest finish time of jobs processed so far)
  - **State merger** = union, min

$$\underbrace{g_i}_{\text{Cost-to-go}}(\underbrace{J_i}_{\text{Jobs remaining}}, \underbrace{f_i}_{\text{Last finish time}}) = \max_{j \in J_i} \left\{ \underbrace{(f_i + p_j - d_j)^+}_{\text{Tardiness of job } j} + g_{i+1}(J_i \setminus \{j\}, f_i + p_j) \right\}$$

Boundary condition:

$$g_{n+1}(\emptyset, f_{n+1}) = 0$$

Optimal value:

$$g_1(\{1, \ldots, n\}, 0)$$

Merger of states in *M* = $\left(\bigcup_{(J_i, f_i) \in M} J_i, \min_{(J_i, f_i) \in M} \{f_i\} \right)$

Dynamic Programming Model

- Single machine scheduling with due dates
  - Easy to **add constraints** that are functions of current state
    - Release times
    - Shutdown periods
    - Precedence constraints on jobs
  - Easy to use **more complicated cost function** that is a function of current state
    - Step functions, etc.
    - Cost that depends on which jobs have been processed.

Dynamic Programming Model

- Scheduling with sequence-dependent setup times
  - **State** = (J_i, last job processed, f_i)
  - **State merger** requires modification of states

$$g_i(J_i, \ell_i, f_i) = \max_{j \in J_i} \left\{ (f_i + p_{\ell_i j} - d_j)^+ + g_{i+1}\big(J_i \setminus \{j\}, j, f_i + p_{\ell_i j}\big) \right\}$$

Last job processed

Tardiness of job j

Processing + setup time

Dynamic Programming Model

- Scheduling with sequence-dependent setup times
  - To allow for **state merger:**
  - **State** = (J_i, set L_i of pairs (ℓ_i, f_i), representing jobs that could have been the last processed)

$$g_i(J_i, L_i) = \max_{j \in J_i} \left\{ \left(\min_{(\ell_i, f_i) \in L_i} \{ f_i + p_{\ell_i j} \} - d_j \right)^+ + g_{i+1}\left(J_i \setminus \{j\}, \left\{ \left(j, \min_{(\ell_i, f_i) \in L_i} \{ f_i + p_{\ell_i j} \} \right) \right\} \right) \right\}$$

Merger of states in *M* = $\left(\bigcup_{(J_i, L_i) \in M} J_i, \bigcup_{(J_i, L_i) \in M} L_i, \right)$

Dynamic Programming Model

- Max cut problem on a graph.
  - Partition nodes into 2 sets so as to maximize total weight of connecting edges.
  - **State** = marginal benefit of placing each remaining vertex on left side of cut.
  - **State merger** =
    - Componentwise min if all components ≥ 0 or all ≤ 0; 0 otherwise
    - Adjust incoming arc weights
- Max 2-SAT.
  - Similar to max cut.

Branching Algorithm

- Solve optimization problem using a novel **branch-and-bound** algorithm.
  - Branch on nodes in **last exact layer** of relaxed decision diagram.
    - …rather than branch on variables.
    - Create a new **relaxed DD rooted** at each branching node.
    - Prune search tree using bounds from relaxed DD.

Bergman, Ciré, van Hoeve, JH (2016)

Branching Algorithm

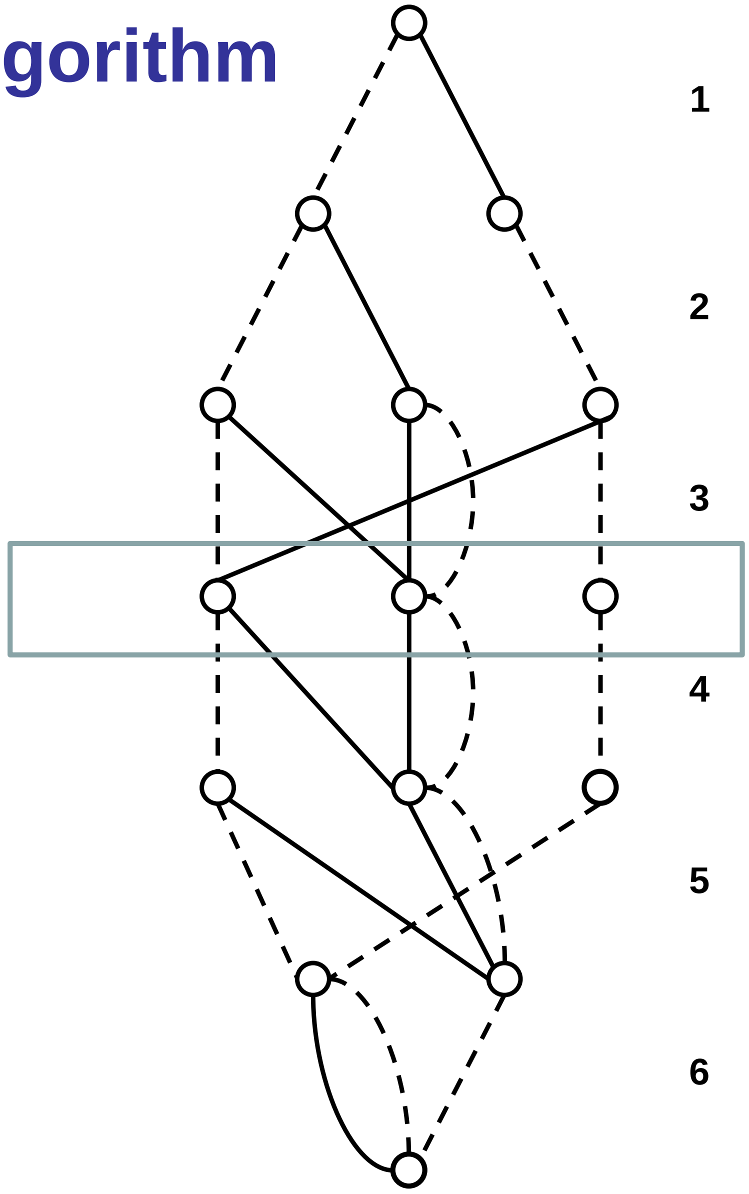
- Solve optimization problem using a novel **branch-and-bound** algorithm.
  - Branch on nodes in **last exact layer** of relaxed decision diagram.
    - …rather than branch on variables.
    - Create a new **relaxed DD rooted** at each branching node.
    - Prune search tree using bounds from relaxed DD.
  - Advantage: a manageable number states may be reachable in first few layers.
    - …even if the state space is **exponential**.
    - Alternative way of dealing with **curse of dimensionality**.

Bergman, Ciré, van Hoeve, JH (2016)

Branching Algorithm

Branching in a relaxed decision diagram

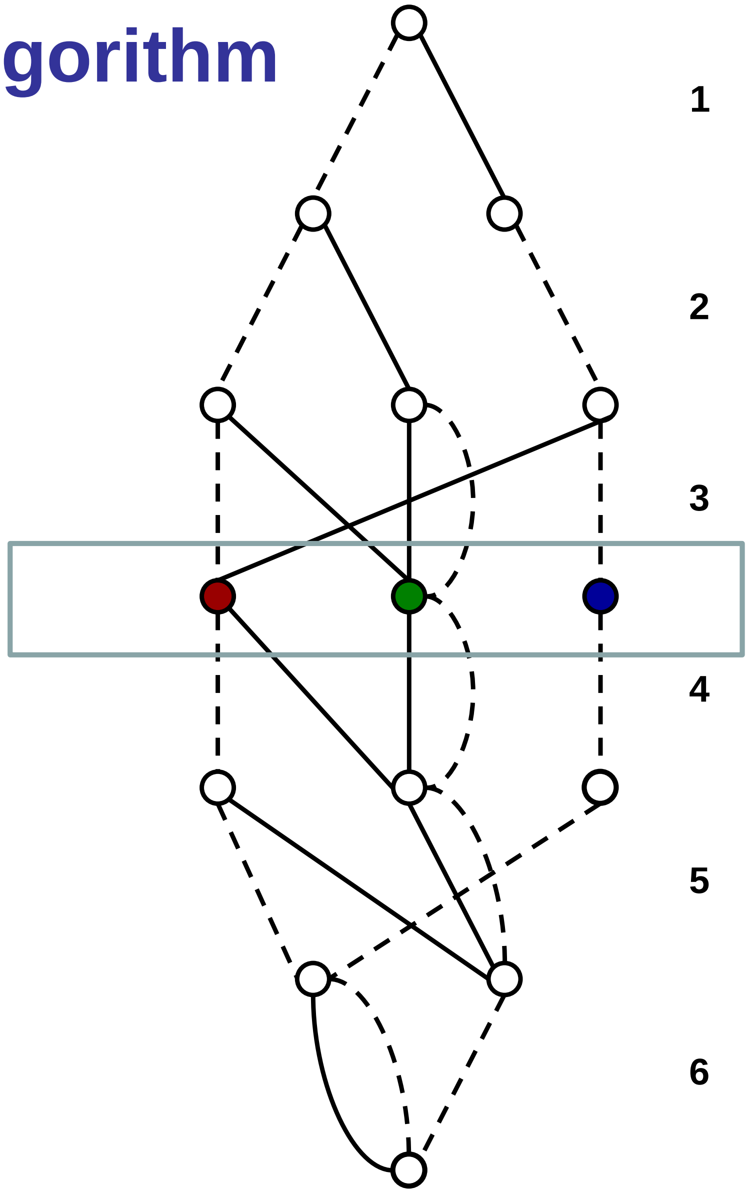
Diagram is exact down to here



Branching Algorithm

Branching in a relaxed
decision diagram

Branch on nodes
in this layer

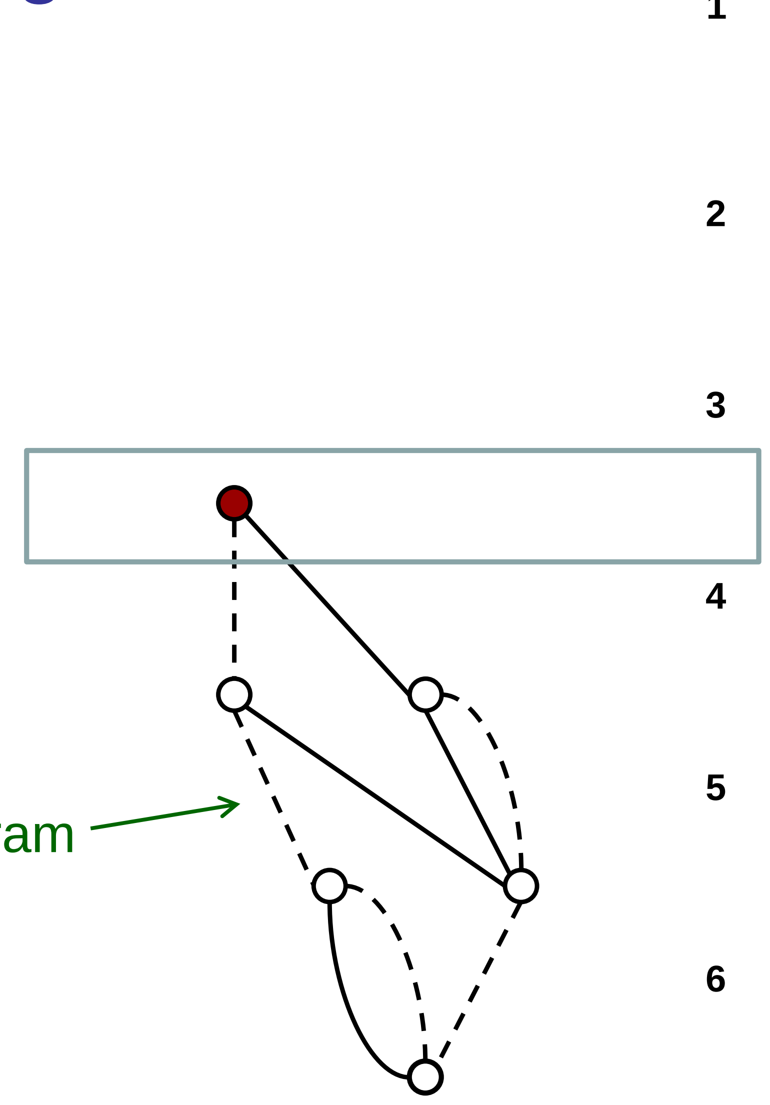


Branching Algorithm

Branching in a relaxed decision diagram

First branch

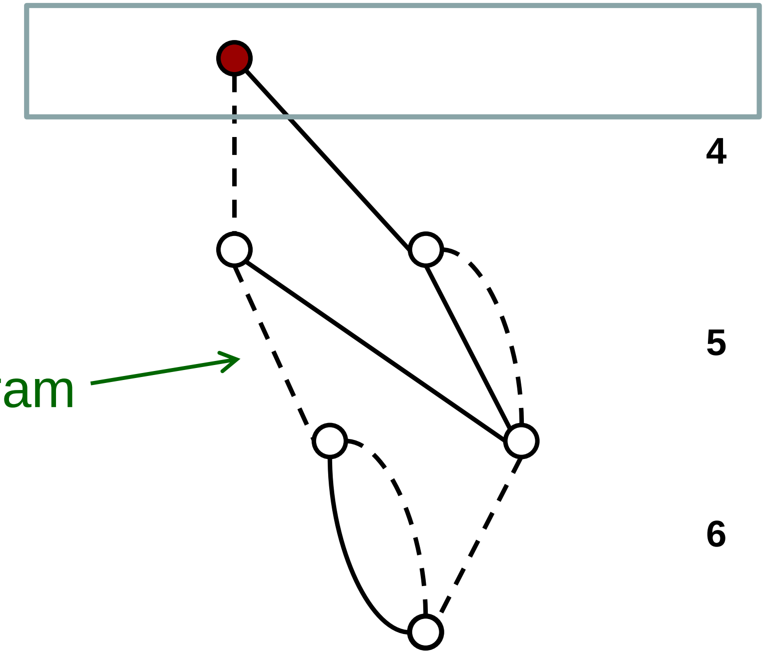
New relaxed decision diagram



Branching Algorithm

Branching in a relaxed decision diagram

First branch



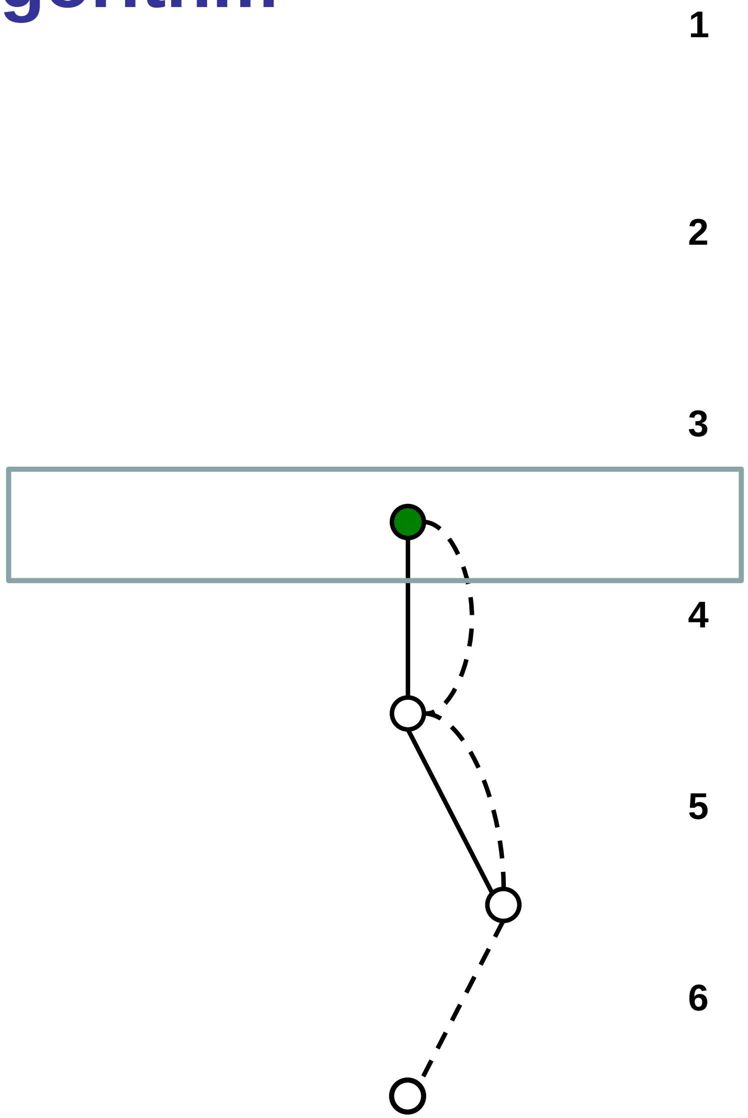
New relaxed decision diagram

Prune this branch if **cost bound** from relaxed DD is **no better** than cost of best feasible solution found so far (**branch and bound**).

Branching Algorithm

Branching in a relaxed
decision diagram

Second branch



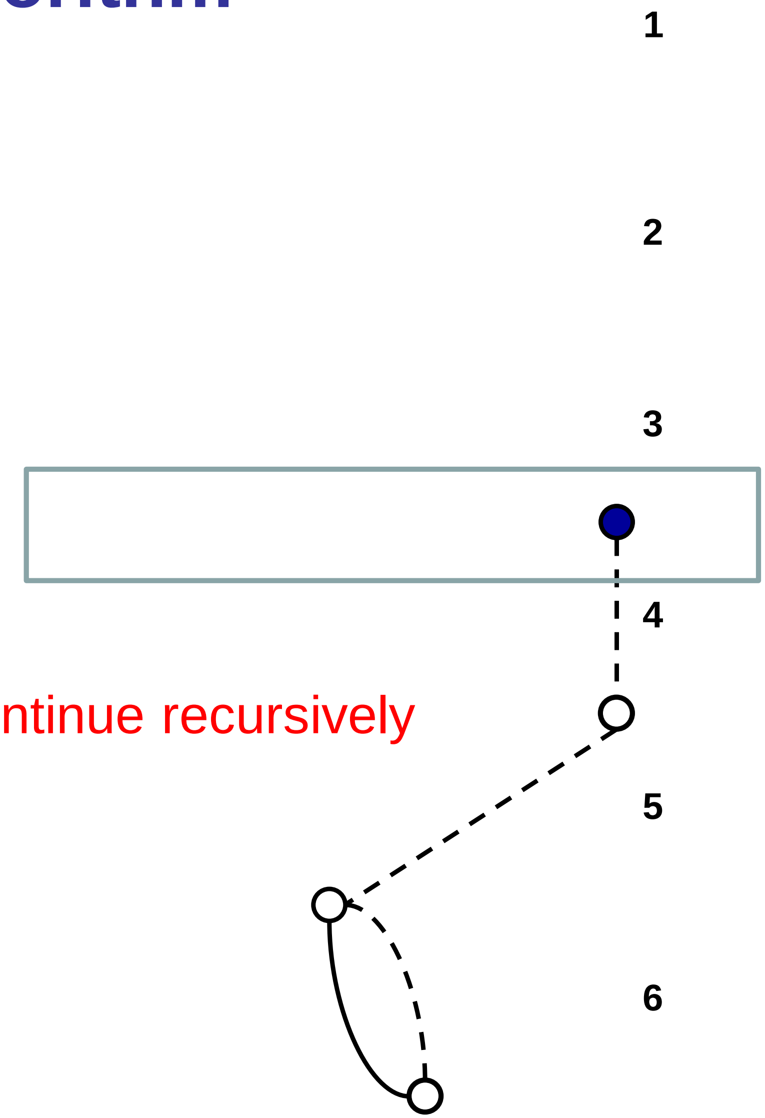
Prune this branch if **cost bound** from relaxed DD is **no better** than cost of best feasible solution found so far (**branch and bound**).

Branching Algorithm

Branching in a relaxed decision diagram

Third branch

Continue recursively



Prune this branch if **cost bound** from relaxed DD is **no better** than cost of best feasible solution found so far (**branch and bound**).

State Space Relaxation?

- This is **very different** from state space relaxation.
  - Problem is **not solved by dynamic programming**.
  - Relaxation created by **merging nodes of DD**
    - …rather than mapping into smaller state space.
  - Relaxation is **constructed dynamically**
    - …as relaxed DD is built.
  - Relaxation uses **same state variables** as exact formulation
    - …which allows branching in relaxed DD

Christofides, Mingozzi, Toth (1981)

Computational performance

- Computational results…
  - Applied to stable set, max cut, max 2-SAT.
    - Superior to commercial MIP solver (CPLEX) on most instances.
    - Obtained best known solution on some max cut instances.
  - Slightly slower than MIP on stable set with precomputed clique cover model, but…

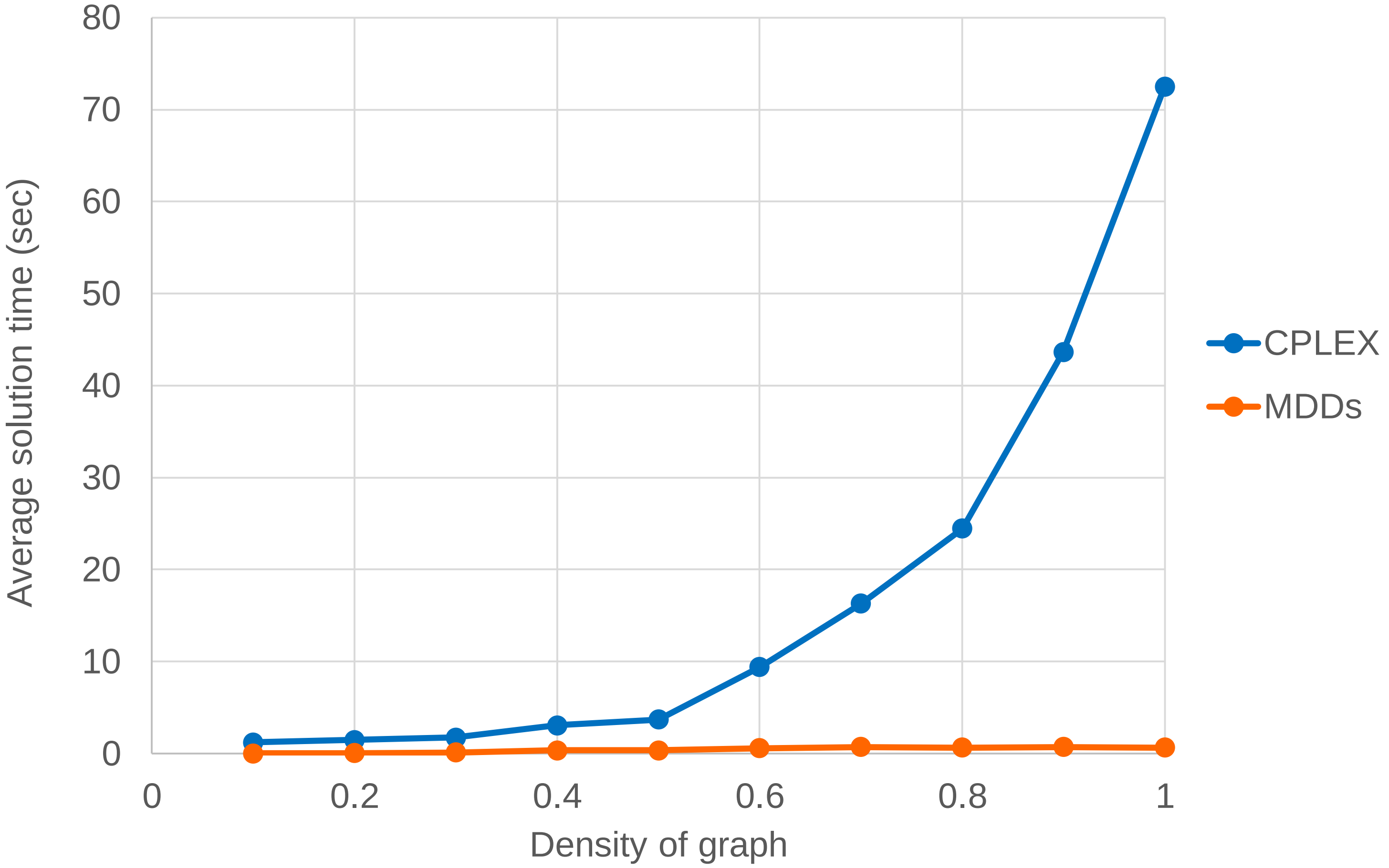
Bergman, Ciré, van Hoeve, JH (2016)

Computational performance

Max cut
on a graph

Avg. solution time
vs
graph density

30 vertices

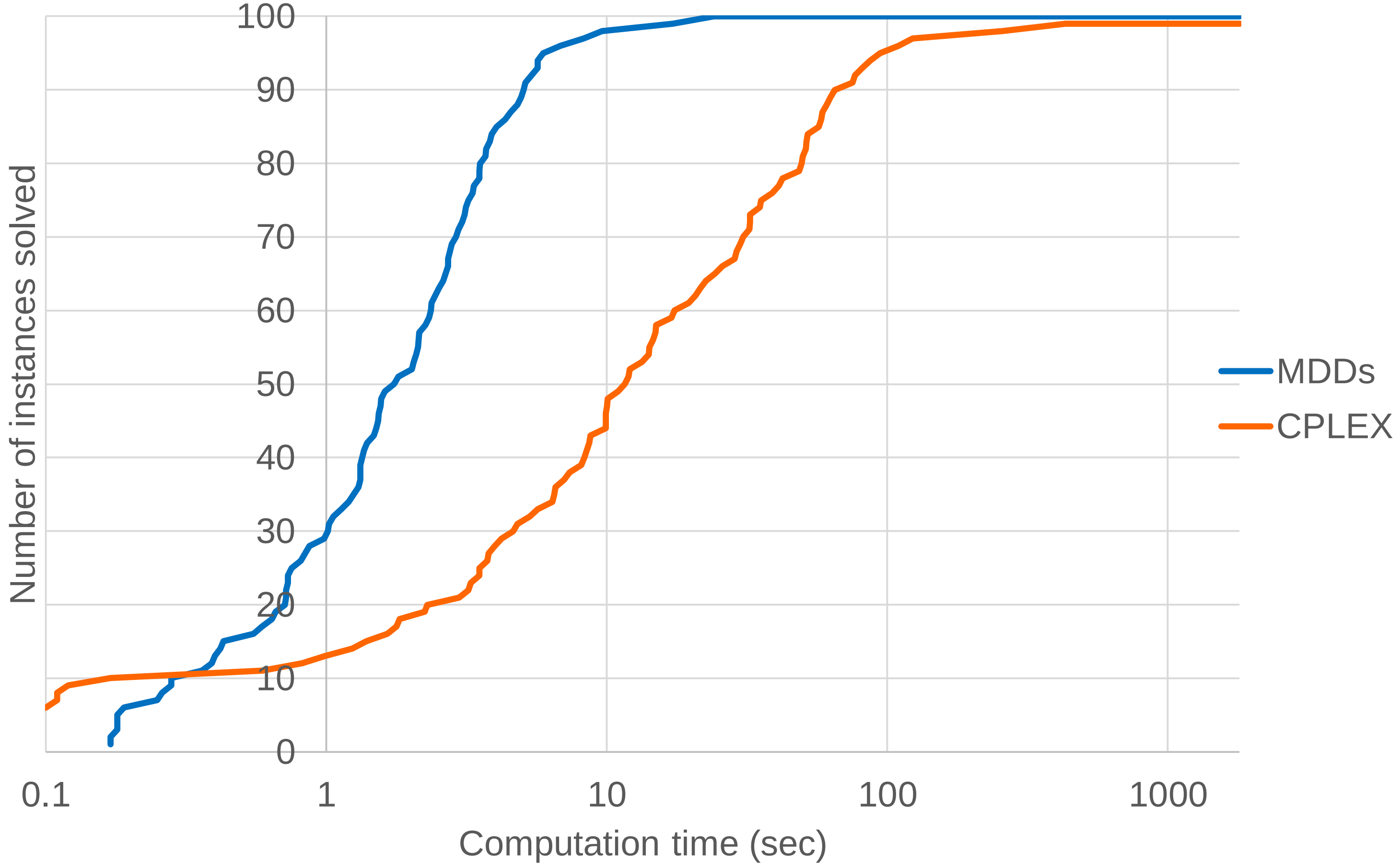


Computational performance

Max 2-SAT

Performance profile

30 variables

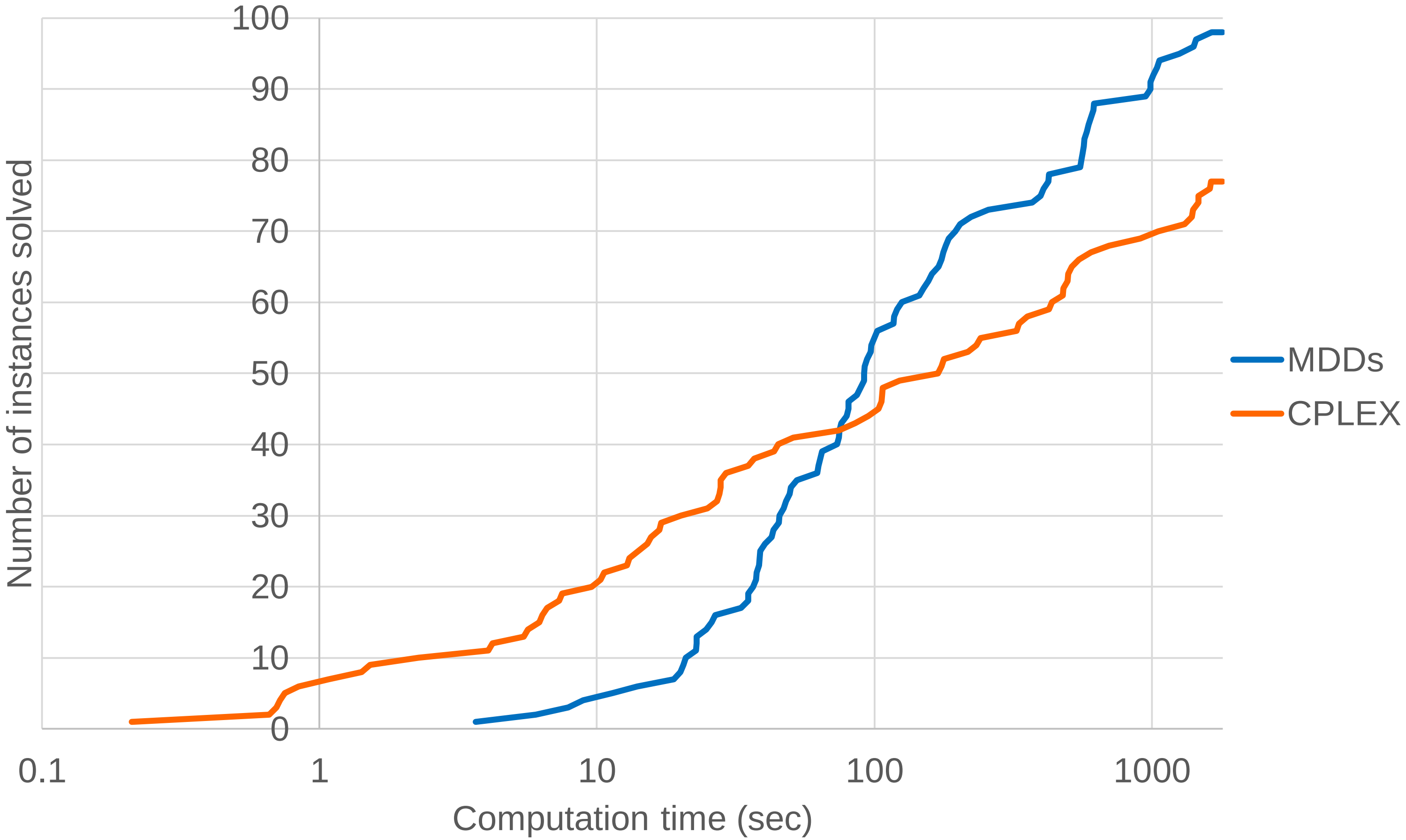


Computational performance

Max 2-SAT

Performance profile

40 variables



Computational performance

- Potential to scale up
  - No need to load large inequality model into solver.
  - **Parallelizes** very effectively
    - **Near-linear** speedup.
    - Much better than mixed integer programming.

Computational performance

- In all computational comparisons so far…
  - Problem is **easily formulated for IP**.
- DD-based optimization is most competitive when…
  - Problem has a recursive **dynamic programming** model…
  - and no convenient IP model.

Computational performance

- In all computational comparisons so far…
  - Problem is **easily formulated for IP**.
- DD-based optimization is most competitive when…
  - Problem has a recursive **dynamic programming** model…
  - and no convenient IP model.
- Such as…
  - Sequencing and scheduling problems (next talk)
  - Planning problems
  - DP problems with exponential state space
    - New approach to “curse of dimensionality”
  - Problems with nonconvex, nonseparable objective function…

Modeling the Objective Function

- Weighted DD can represent **any** objective function
  - Separable functions are the easiest, but any nonseparable function is possible.
  - Can be nonlinear, nonconvex, etc.
  - The issue is complexity of resulting DD

Modeling the Objective Function

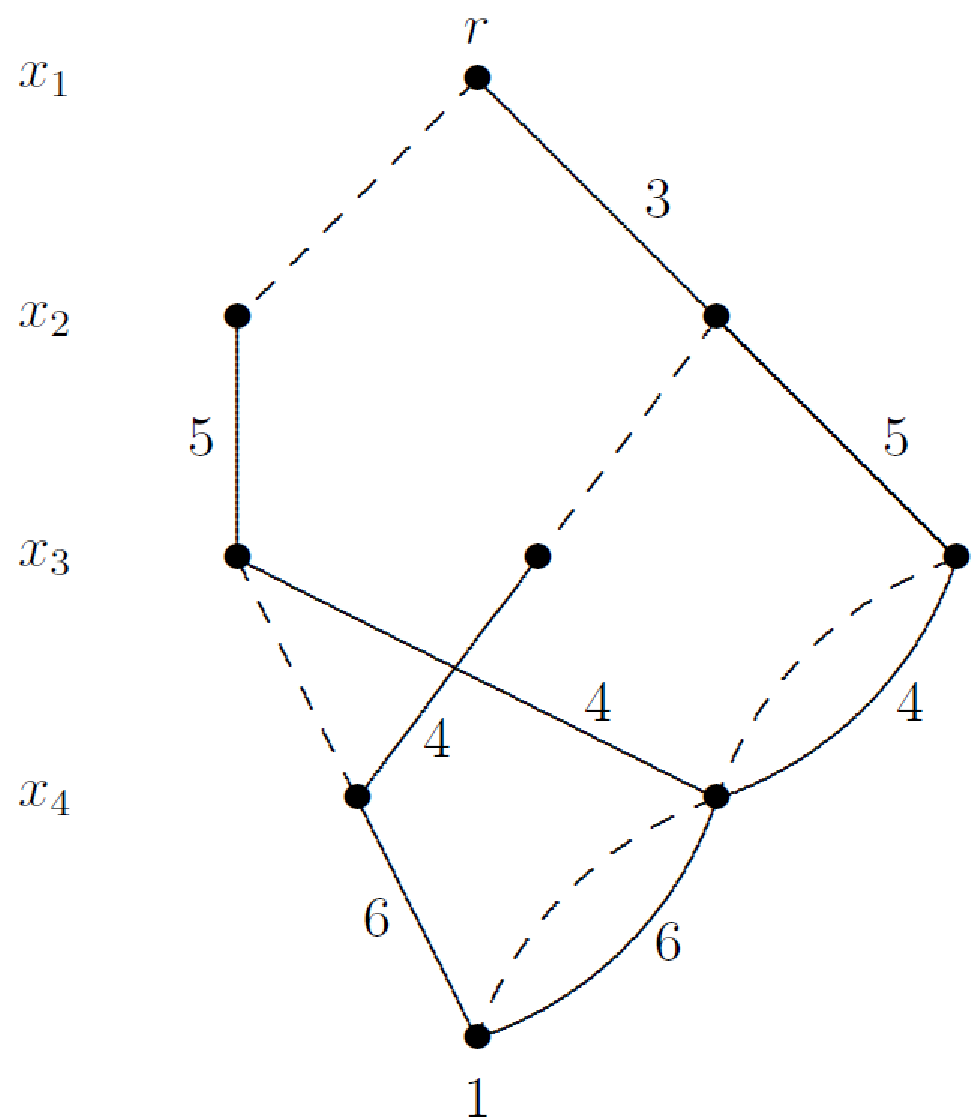
- Weighted DD can represent **any** objective function
  - Separable functions are the easiest, but any nonseparable function is possible.
  - Can be nonlinear, nonconvex, etc.
  - The issue is complexity of resulting DD
- Multiple encodings
  - A given objective function can be encoded by **multiple** assignments of costs to arcs.
  - There is a **unique canonical** arc cost assignment.
    - Which can **reduce size** of exact DD.
  - Design state variables accordingly

Modeling the Objective Function

Set covering with separable cost function

Easy. Just label arcs with weights.

	Set i 1	2	3	4
A	●	●		
B	●		●	●
C		●	●	
D		●		●
Weight	3	5	4	6



$x_i = 1$ when we select set i

Modeling the Objective Function

Nonseparable cost function

Now what?

x	$f(x)$
(0,1,0,1)	6
(0,1,1,0)	7
(0,1,1,1)	8
(1,0,1,1)	5
(1,1,0,0)	6
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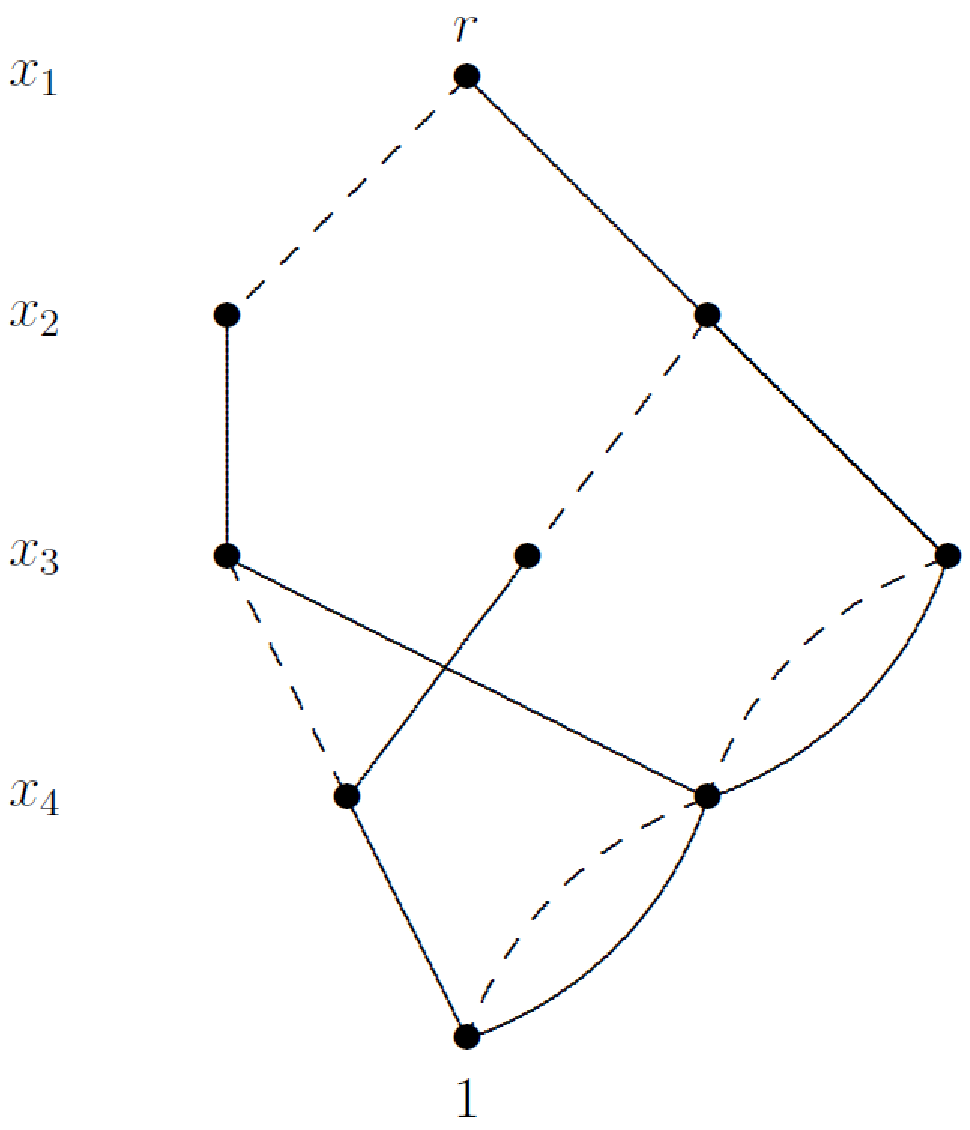
(1,1,1,0)	7
(1,1,1,1)	9



Modeling the Objective Function

Nonseparable cost function

Put costs on leaves of branching tree.

x	$f(x)$
(0,1,0,1)	6
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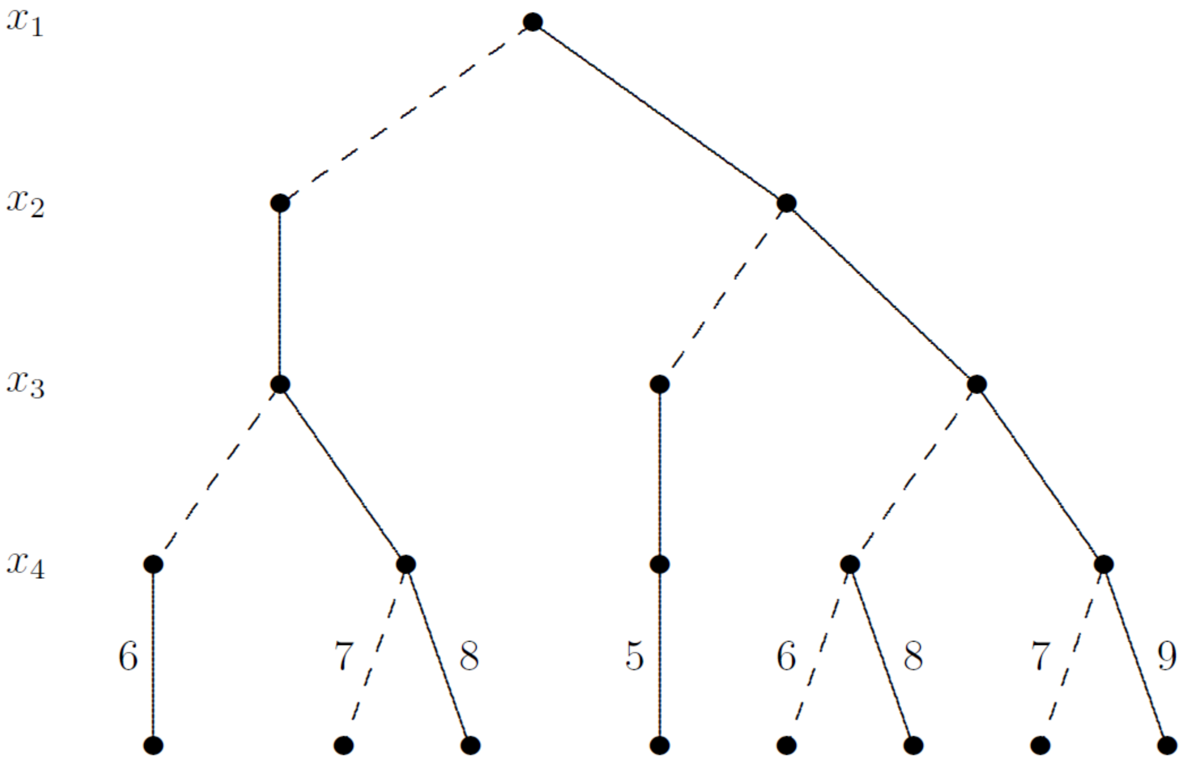
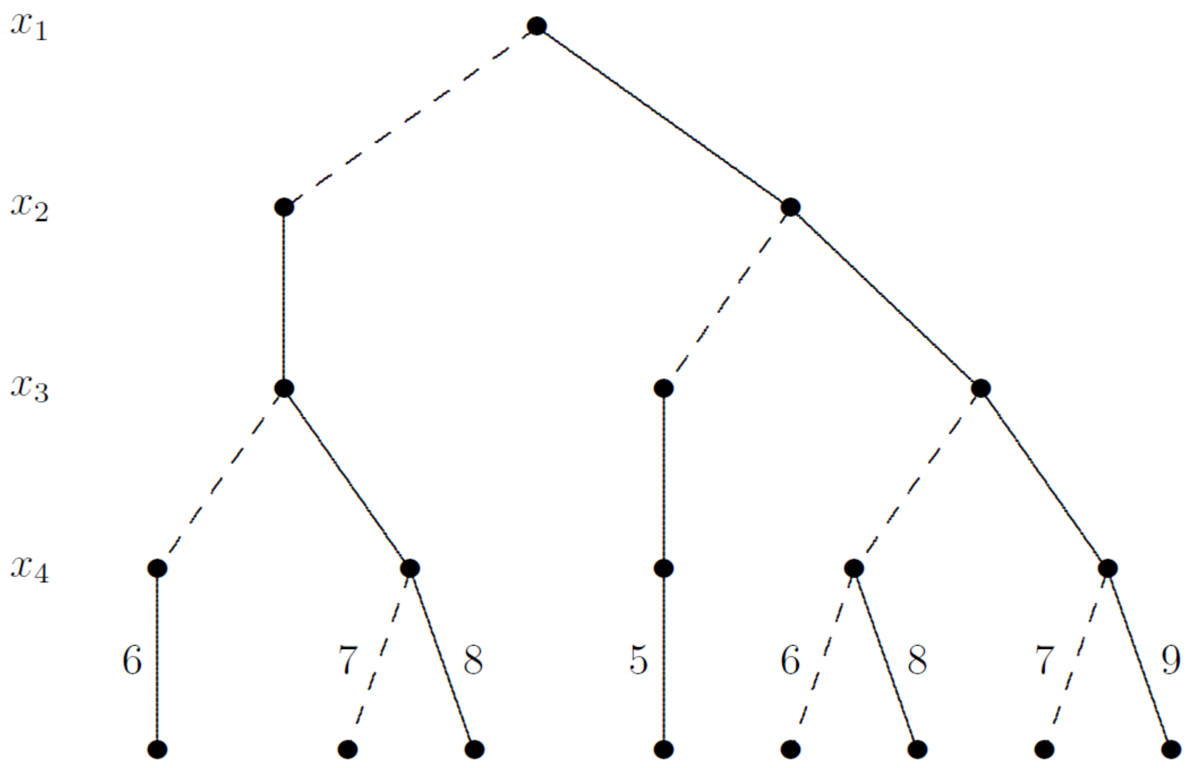


Modeling the Objective Function

Nonseparable cost function

Put costs on leaves of branching tree.

But now we can't reduce the tree to an efficient decision diagram.



Modeling the Objective Function

Nonseparable cost function

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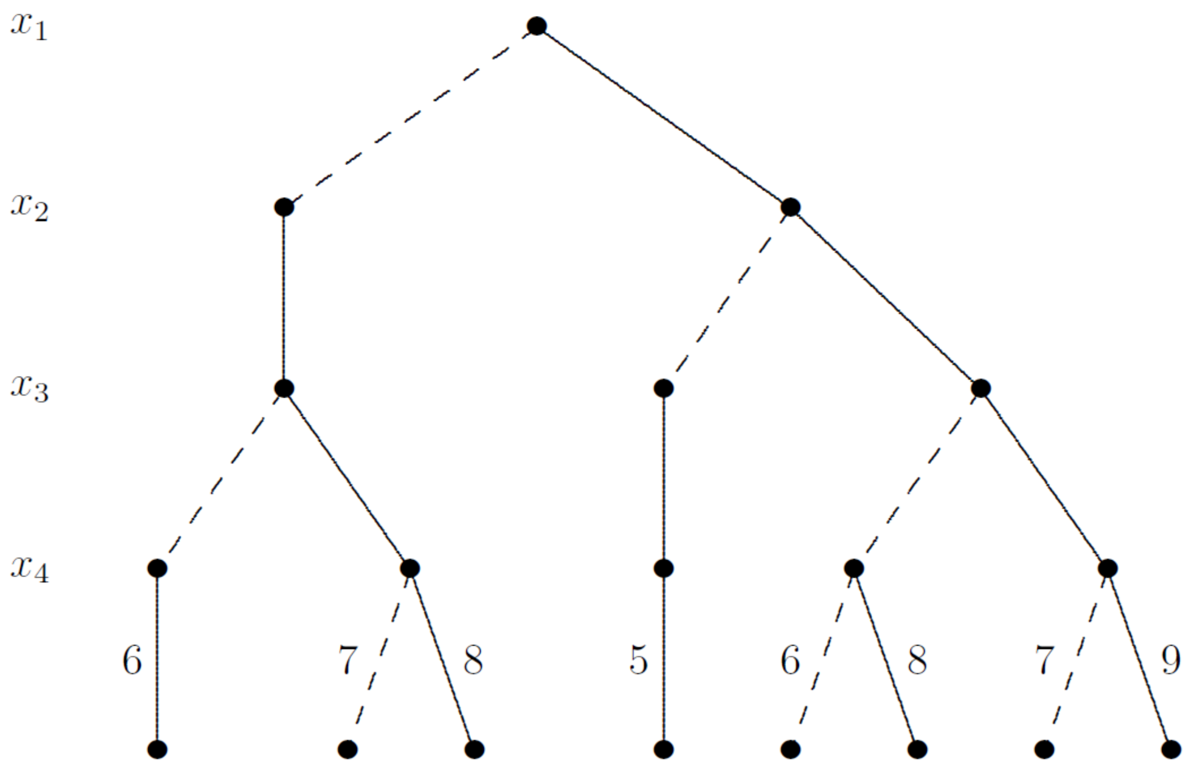
We will rearrange costs to obtain **canonical costs**.



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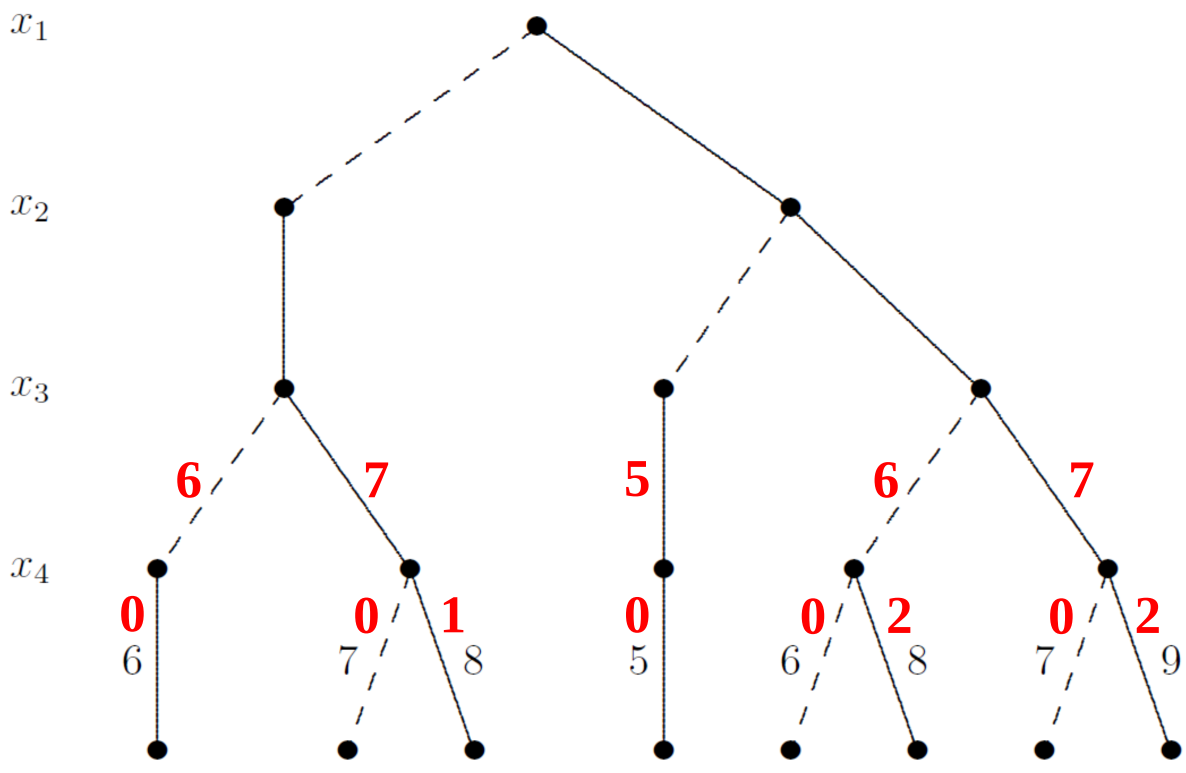

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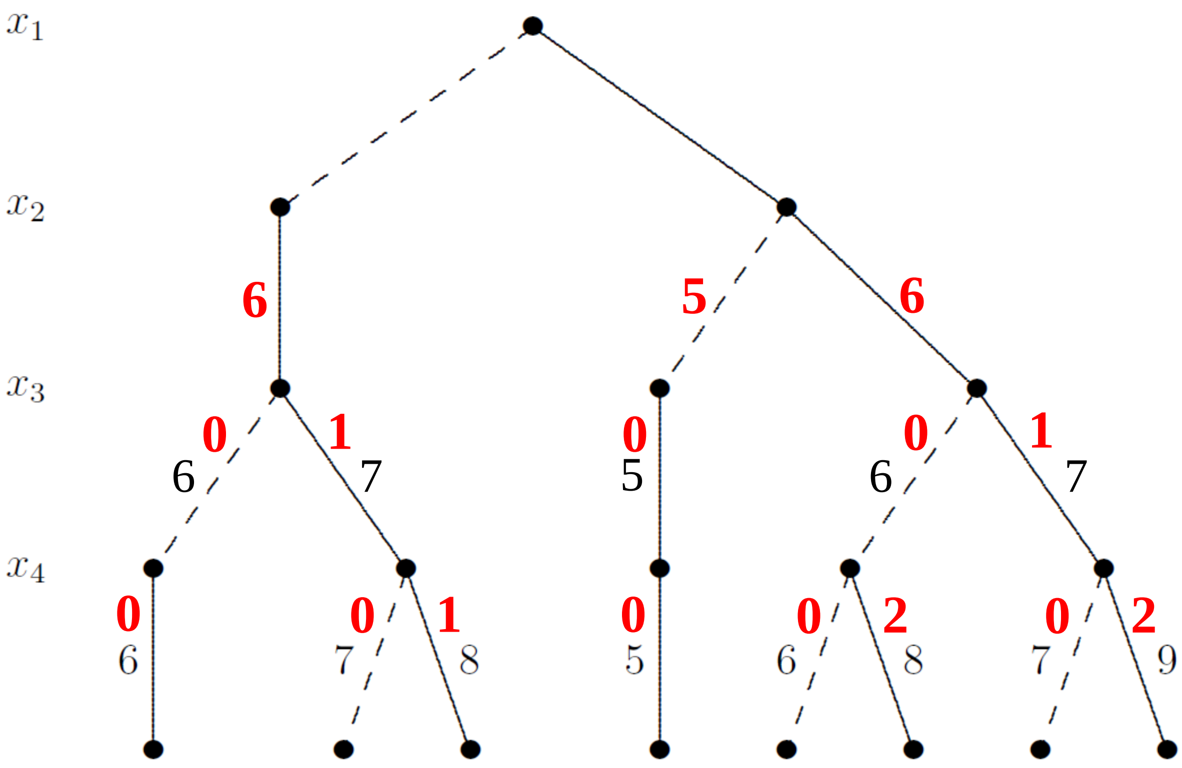

Modeling the Objective Function

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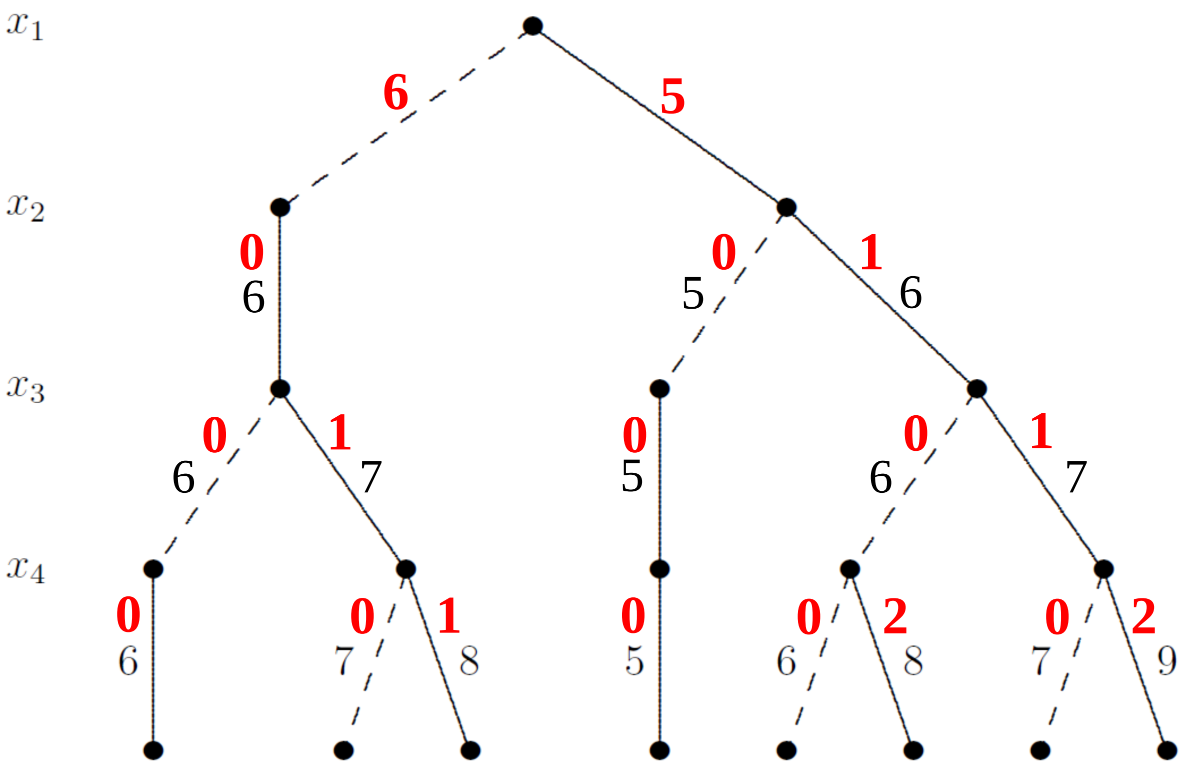
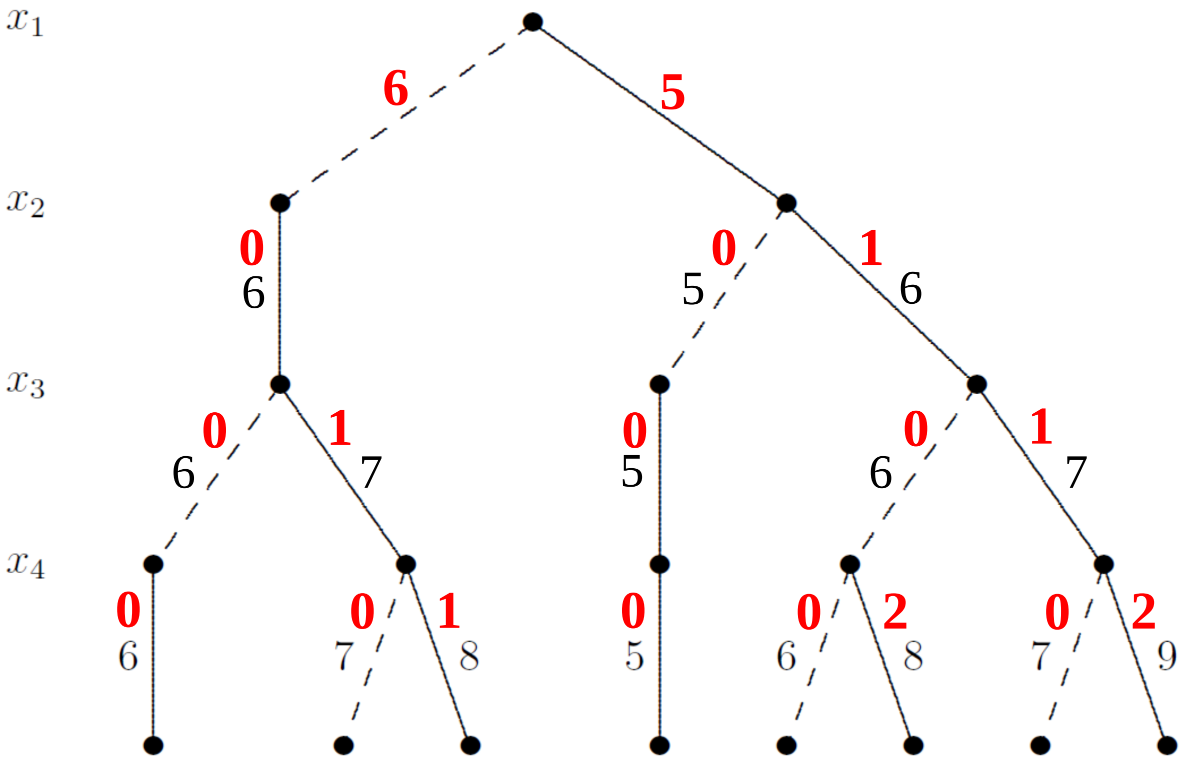


Modeling the Objective Function

Nonseparable cost function

Now the tree can be reduced.



Modeling the Objective Function

Nonseparable cost function

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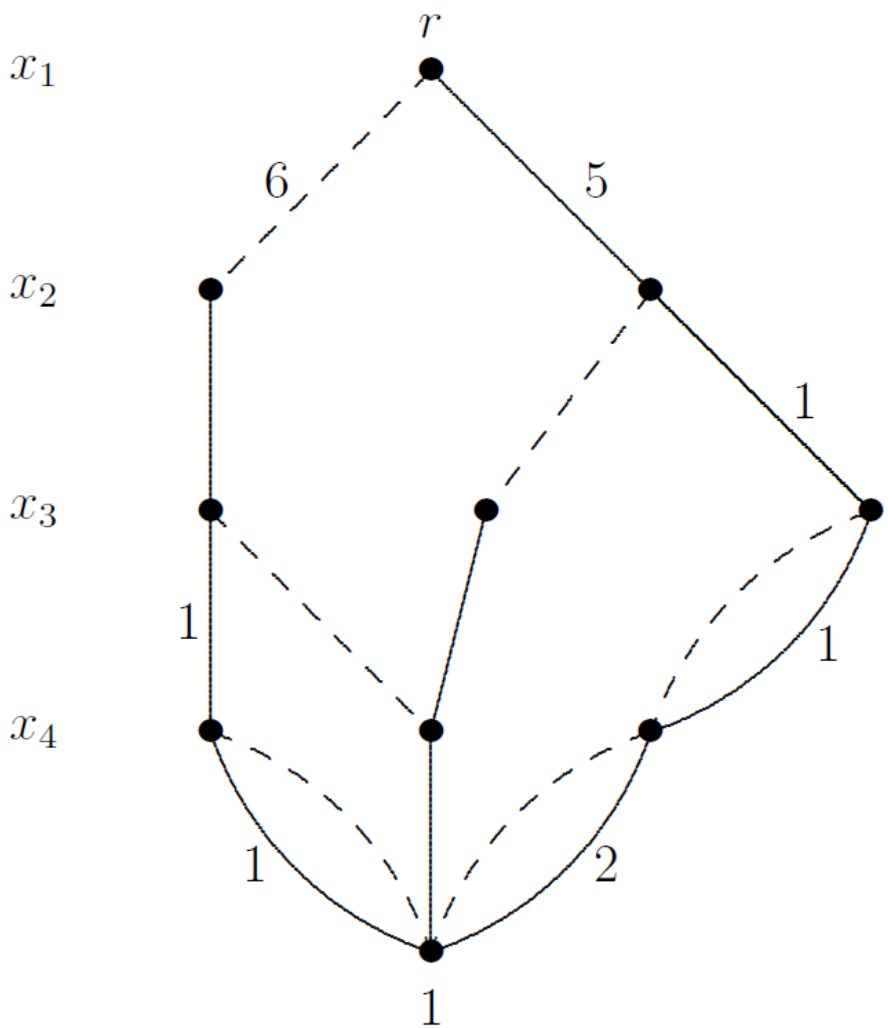
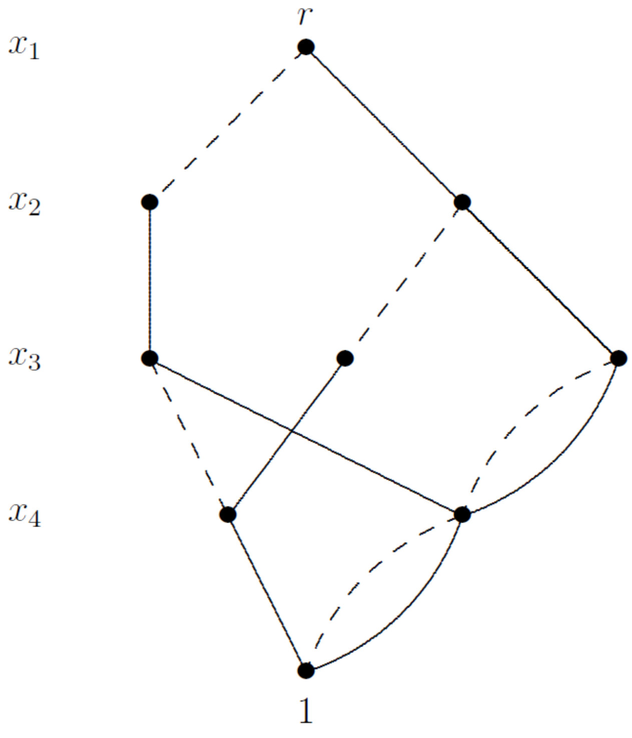
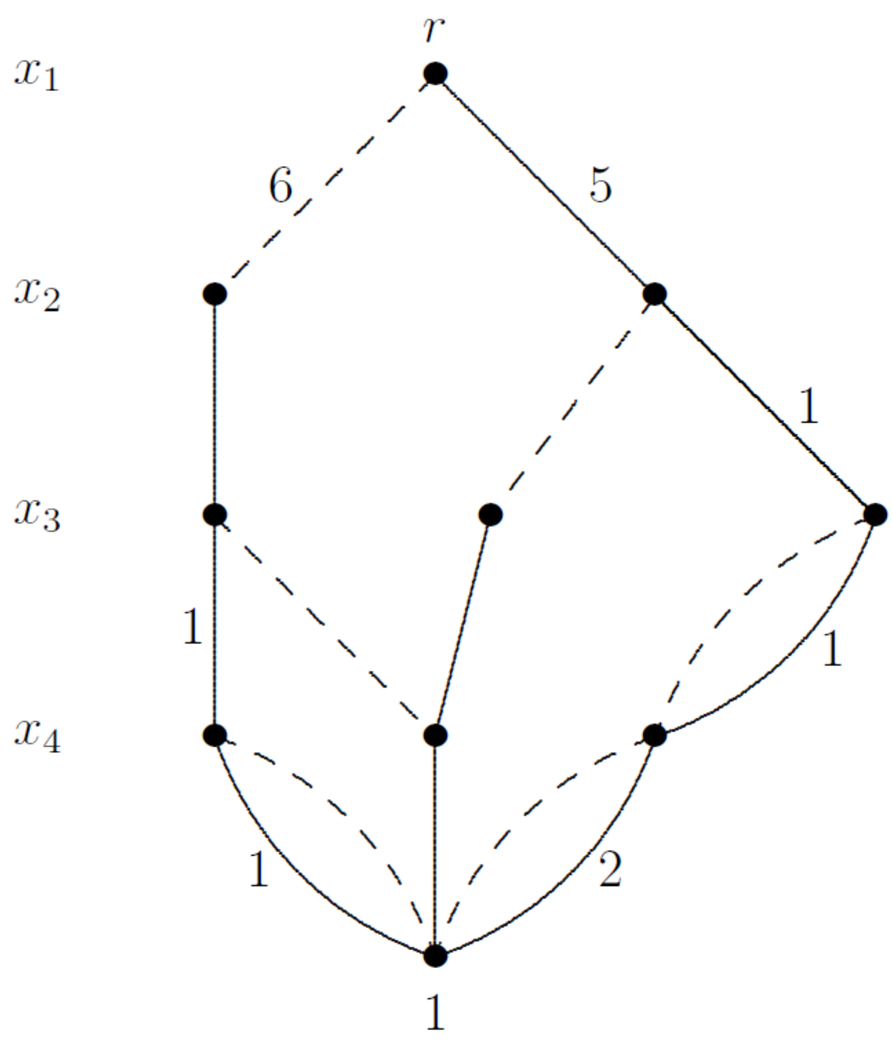


Modeling the Objective Function

Nonseparable cost function

DD is larger than
reduced unweighted DD,
but still compact.





Modeling the Objective Function

Theorem. For a given variable ordering, a given objective function is represented by a **unique** weighted decision diagram with canonical costs.

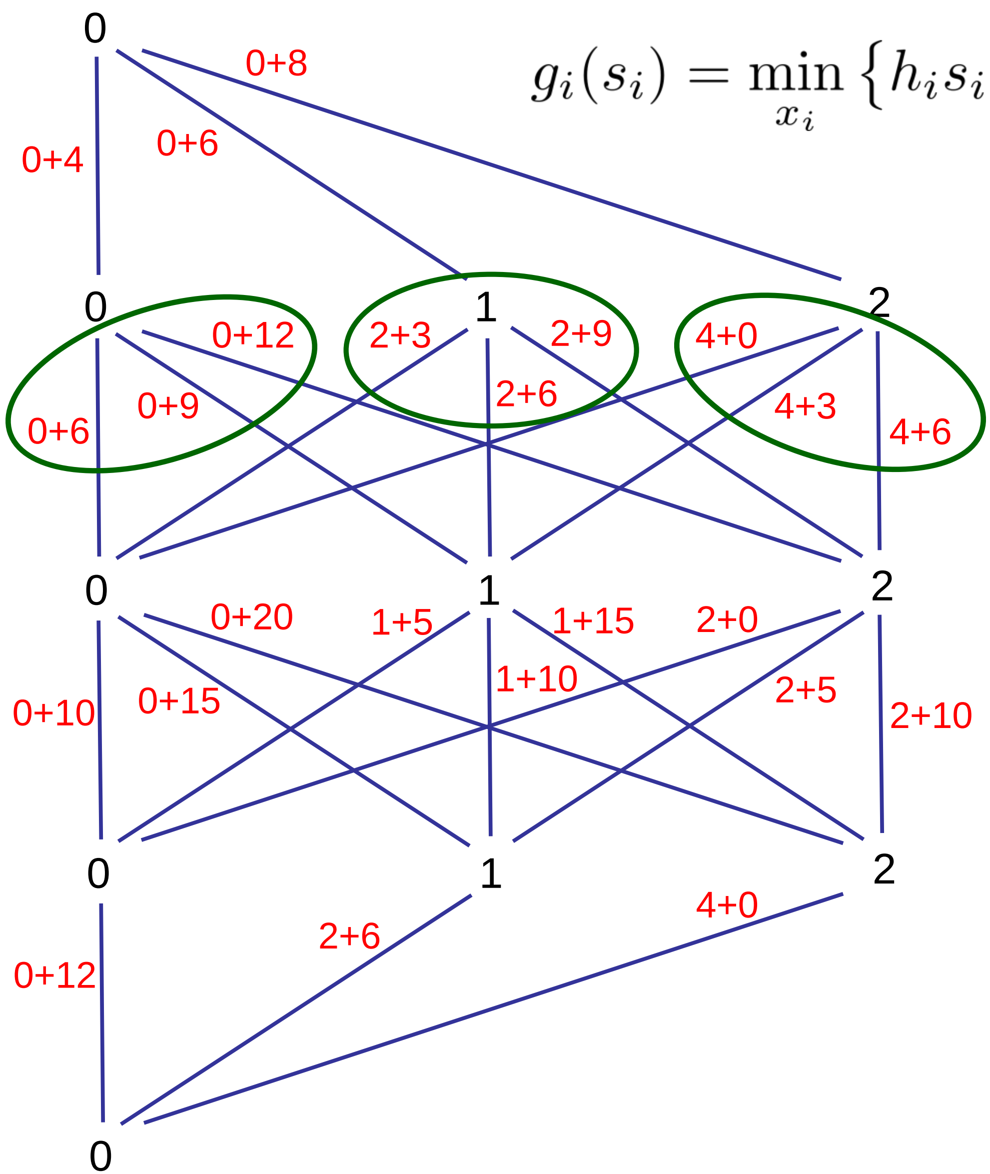
JH (2013),
Similar result for AADDs:
Sanner & McAllester (2005)

Inventory Management Example

- In each period i, we have:
  - Demand d_i
  - Unit production cost c_i
  - Warehouse space m
  - Unit holding cost h_i
- In each period, we decide:
  - Production level x_i
  - Stock level s_i
- Objective:
  - Meet demand each period while minimizing production and holding costs.

Reducing the Transition Graph

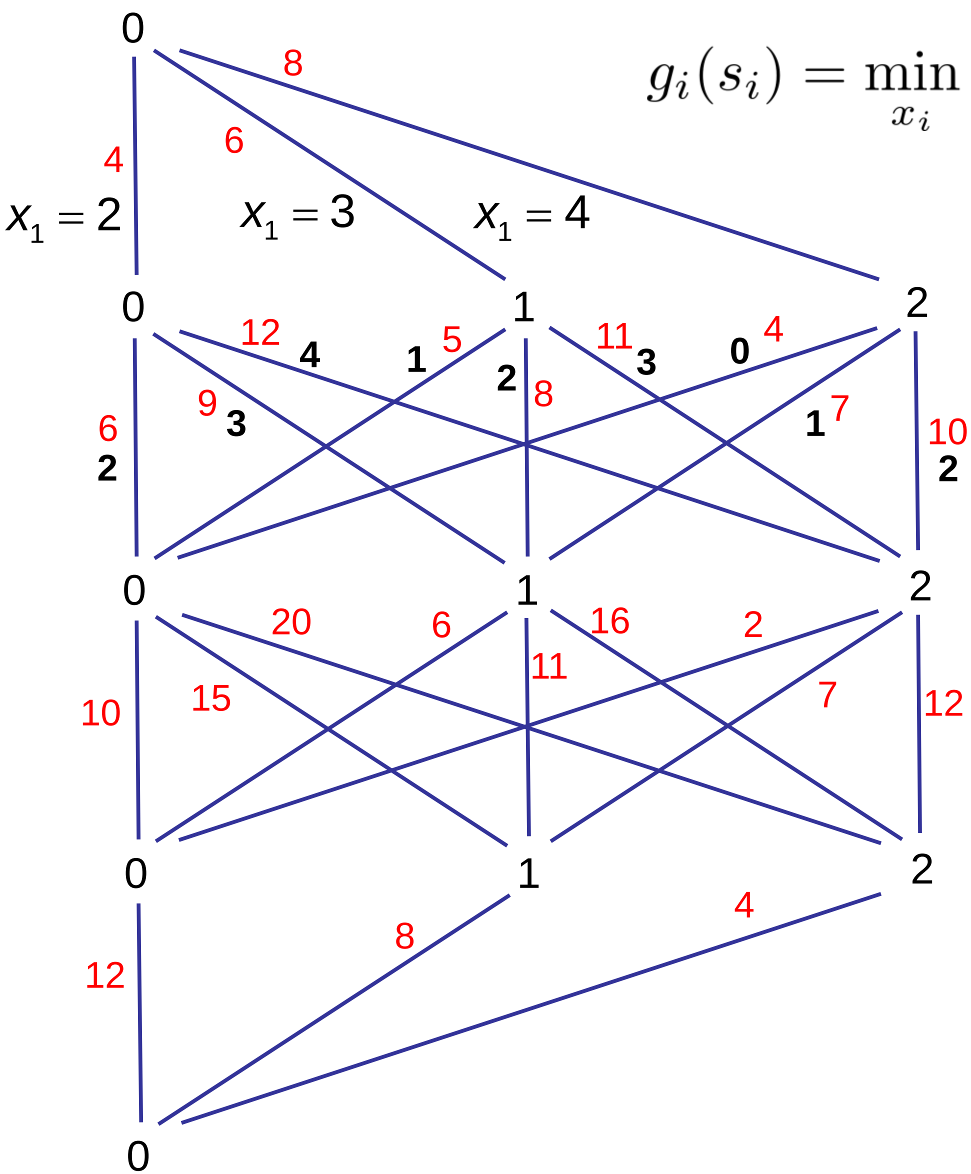
$$g_i(s_i) = \min_{x_i} \left\{ h_i s_i + c_i x_i + g_{i+1}(s_i + x_i - d_i) \right\}$$



Arcs leaving each node are very similar.

- Transition to the same states.
- Have the same costs, up to an offset.

Inventory Problem



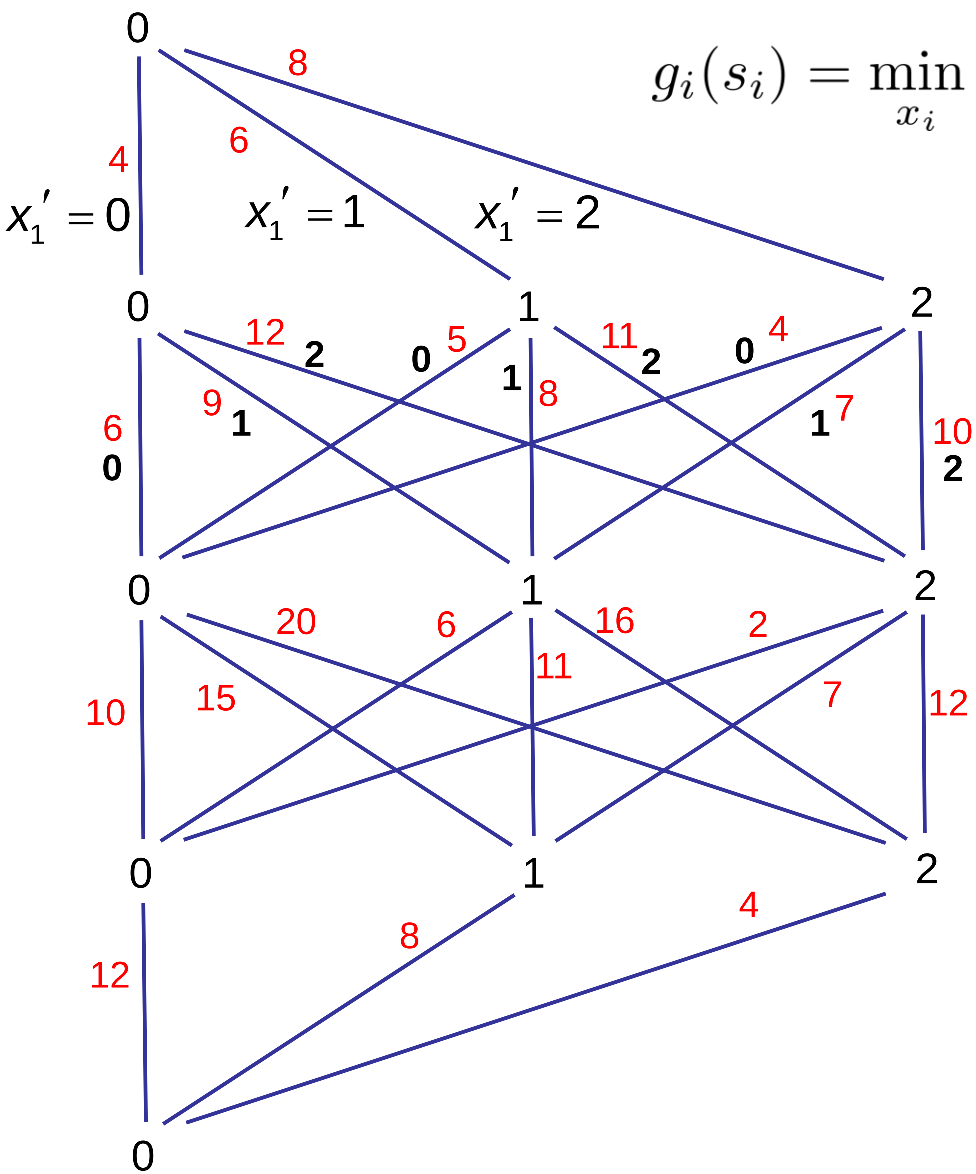
$$g_i(s_i) = \min_{x_i} \left\{ h_i s_i + c_i x_i + g_{i+1}(s_i + x_i - d_i) \right\}$$

To equalize controls, let

$$x_i' = s_i + x_i - d_i$$

be the stock level in next period.

Inventory Problem



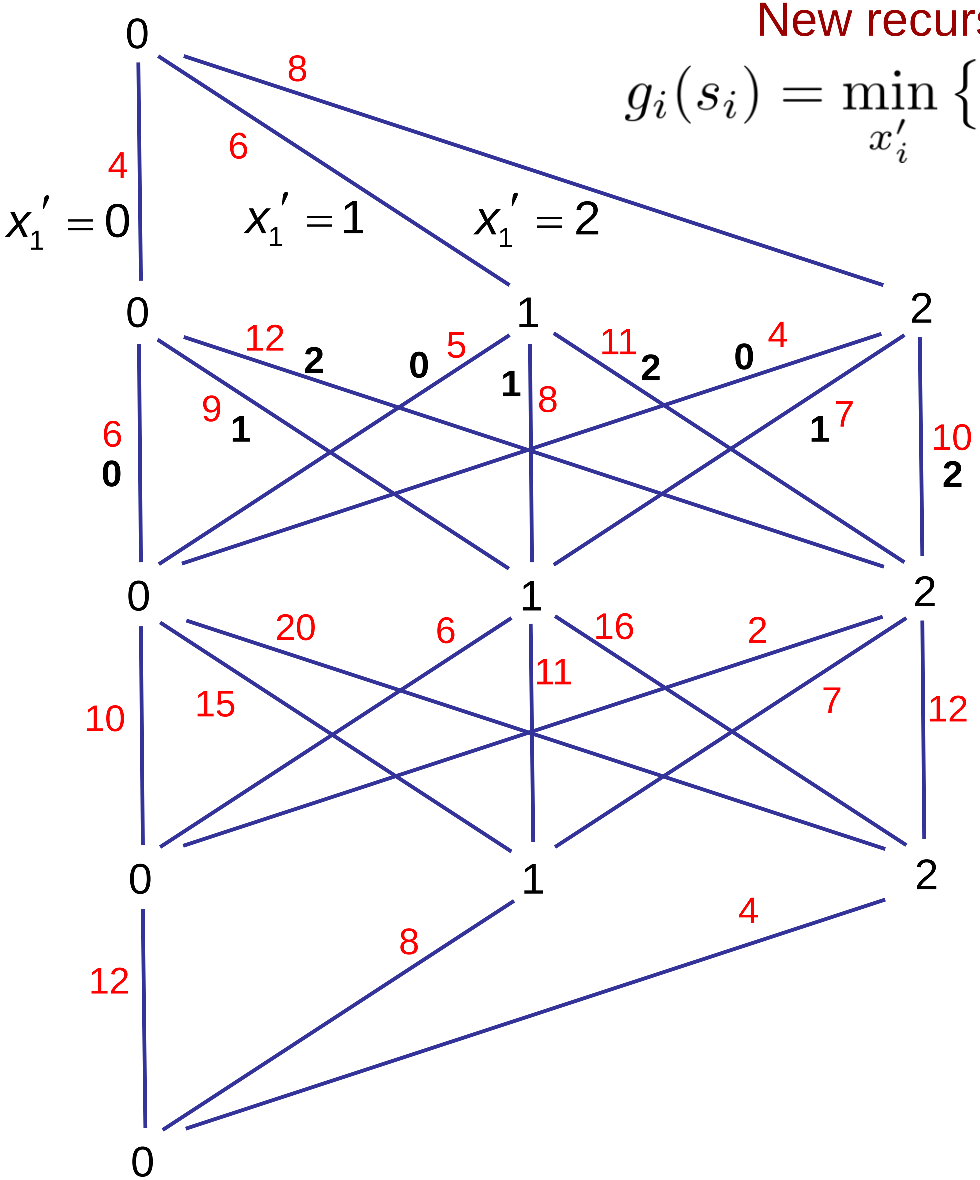
$$g_i(s_i) = \min_{x_i} \left\{ h_i s_i + c_i x_i + g_{i+1}(s_i + x_i - d_i) \right\}$$

To equalize controls, let

$$x_i' = s_i + x_i - d_i$$

Be the stock level in next period.

Inventory Problem



New recursion:

$$g_i(s_i) = \min_{x_i'} \left\{ h_i s_i + c_i(x_i' - s_i + d_i) + g_{i+1}(x_i') \right\}$$

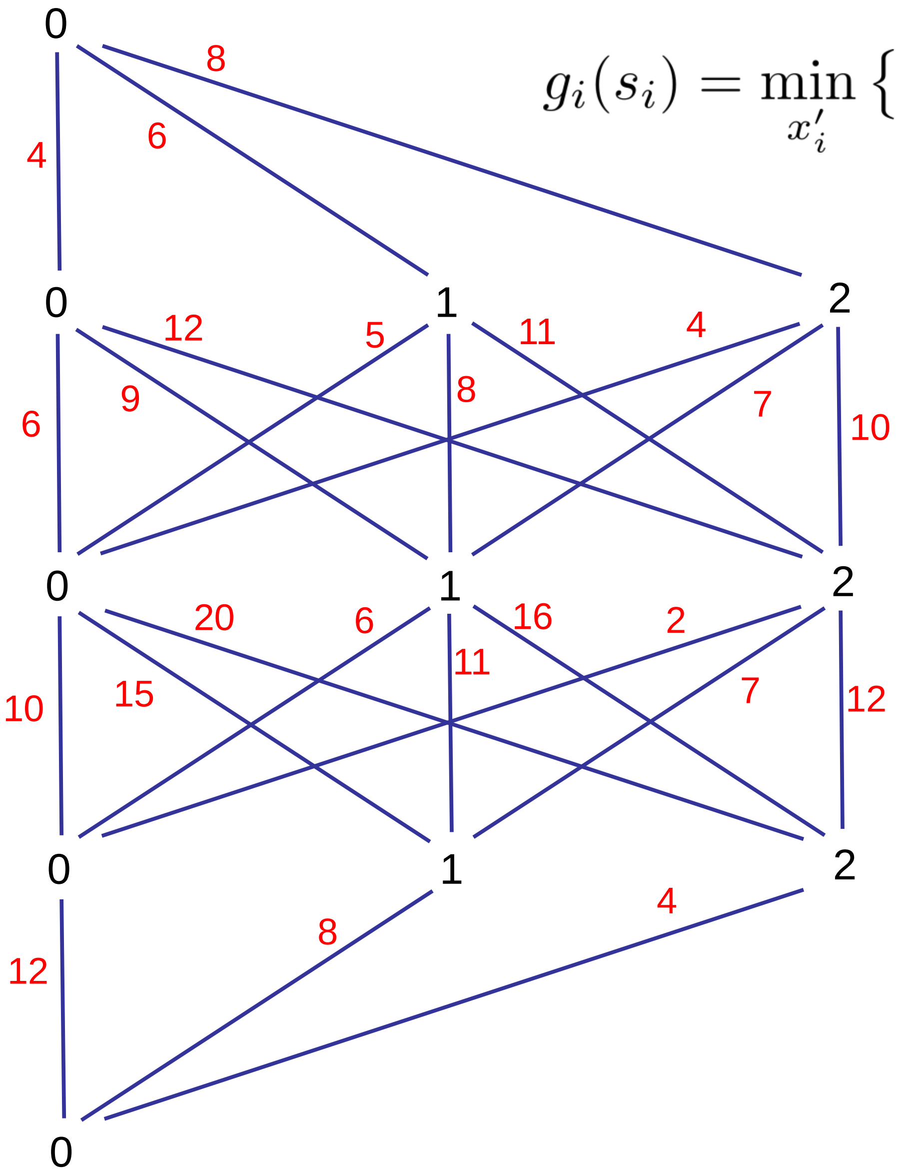
To equalize controls, let

$$x_i' = s_i + x_i - d_i$$

Be the stock level in next period.

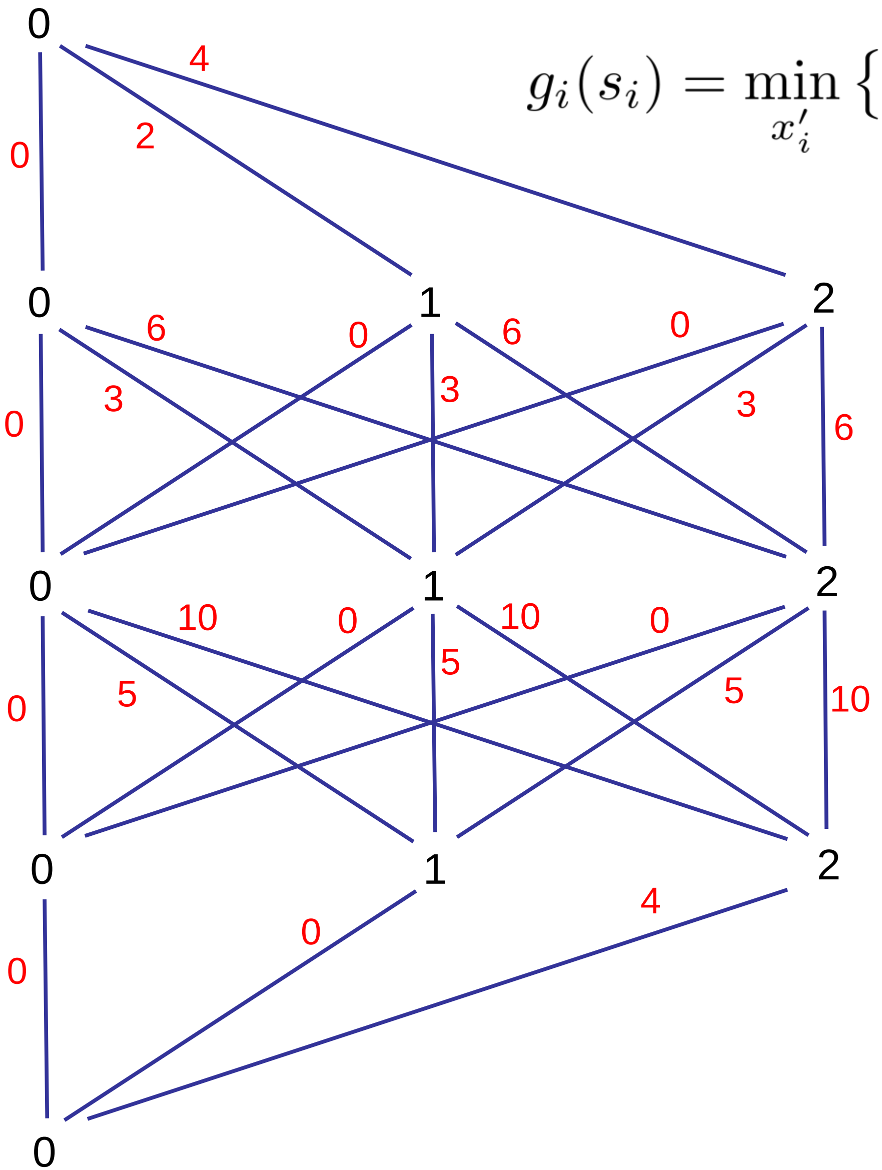
Inventory Problem

$$g_i(s_i) = \min_{x_i'} \left\{ h_i s_i + c_i(x_i' - s_i + d_i) + g_{i+1}(x_i') \right\}$$



To obtain canonical costs, subtract $c_i(m - s_i) + h_i s_i$ from cost on each arc (s_i,s_{i+1}).

Inventory Problem

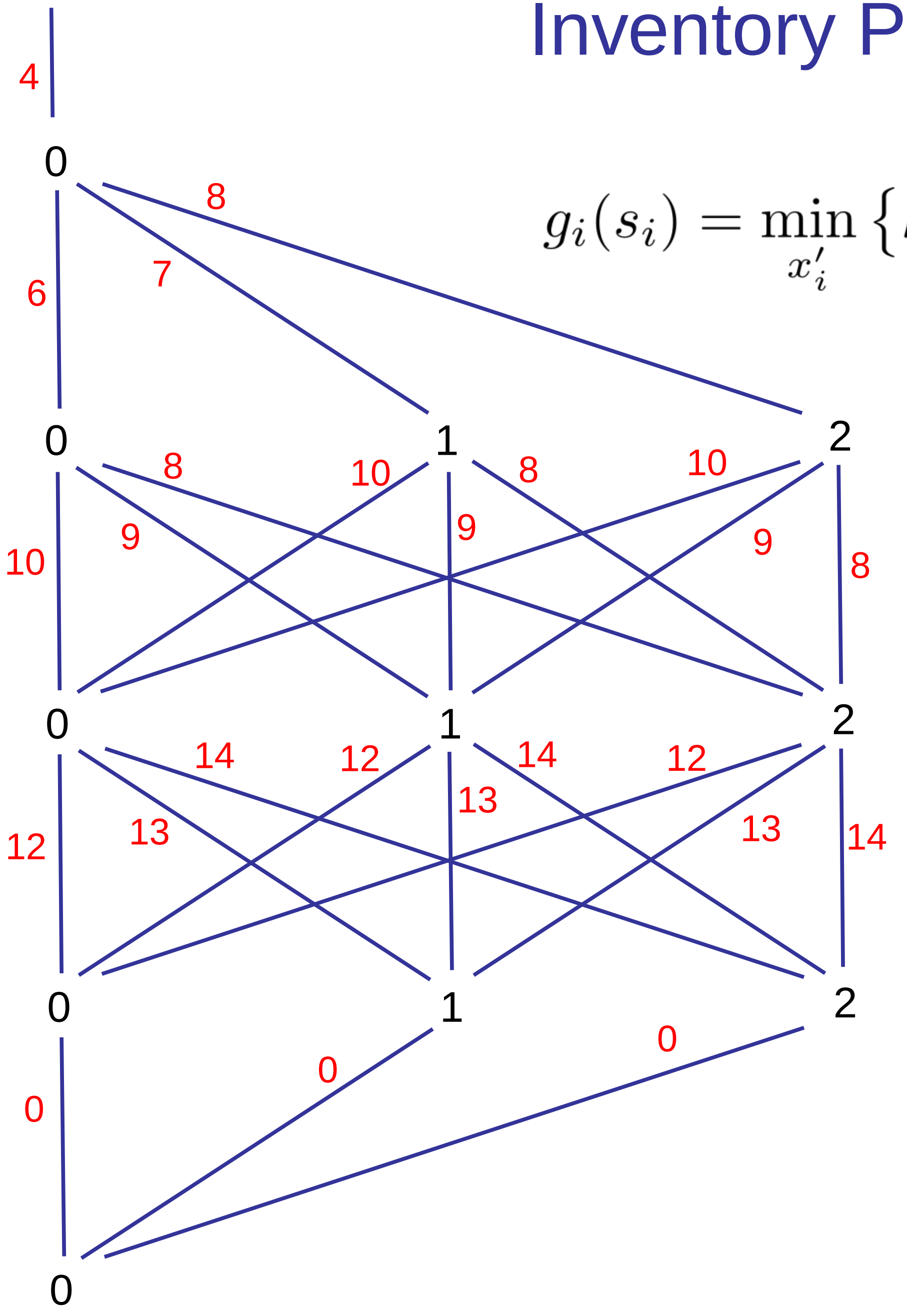


$$g_i(s_i) = \min_{x_i'} \left\{ h_i s_i + c_i(x_i' - s_i + d_i) + g_{i+1}(x_i') \right\}$$

To obtain canonical costs, subtract $c_i(m - s_i) + h_i s_i$ from cost on each arc (s_i, s_{i+1}).

Add these offsets to incoming arcs.

Inventory Problem



$$g_i(s_i) = \min_{x_i'} \left\{ h_i s_i + c_i(x_i' - s_i + d_i) + g_{i+1}(x_i') \right\}$$

To obtain canonical costs, subtract $c_i(m - s_i) + h_i s_i$ from cost on each arc (s_i,s_{i+1}).

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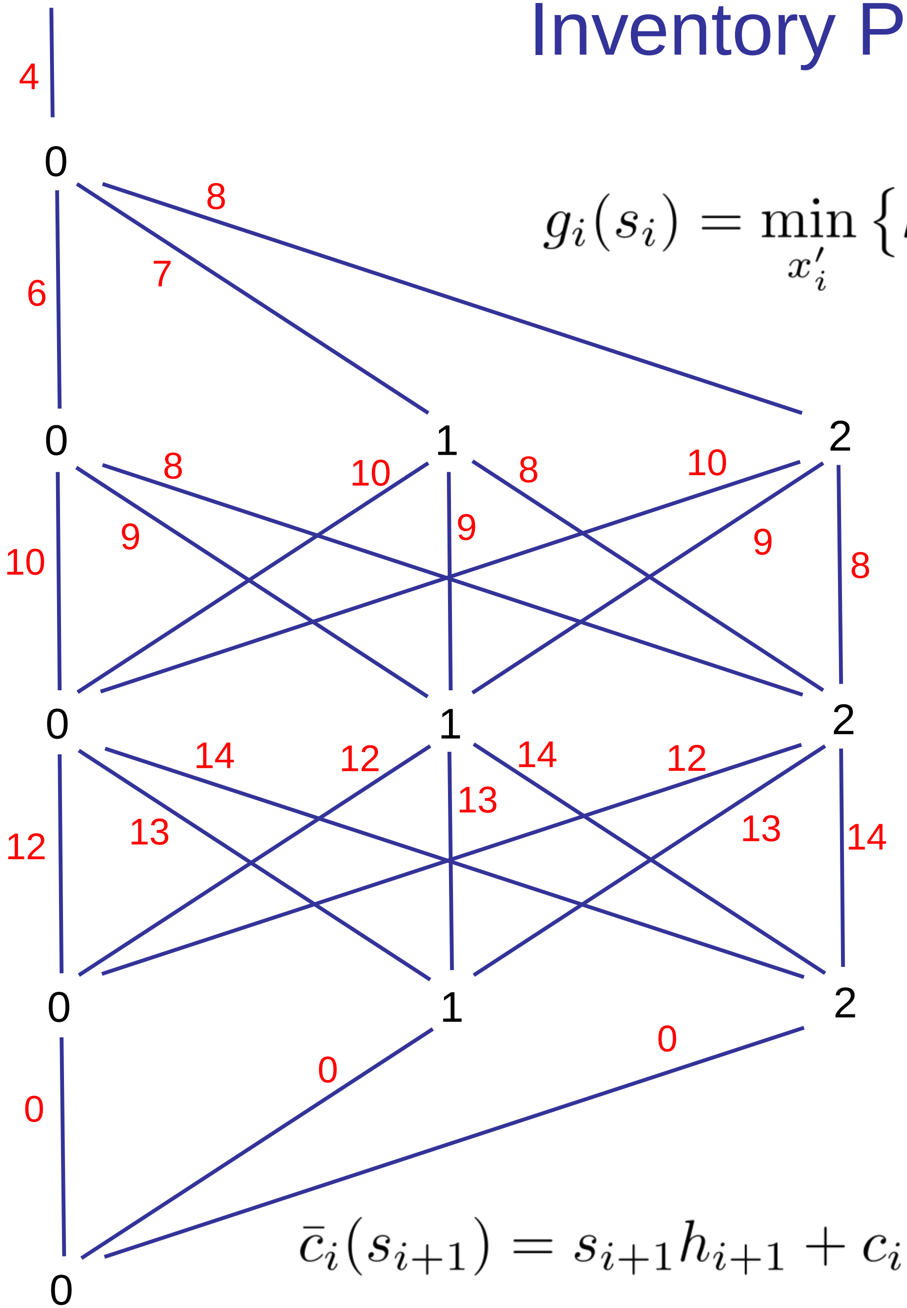
Inventory Problem



$$g_i(s_i) = \min_{x_i'} \left\{ h_i s_i + c_i(x_i' - s_i + d_i) + g_{i+1}(x_i') \right\}$$

To obtain canonical costs, subtract $c_i(m - s_i) + h_i s_i$ from cost on each arc (s_i, s_{i+1}).

Add these offsets to incoming arcs.

Now outgoing arcs look alike.

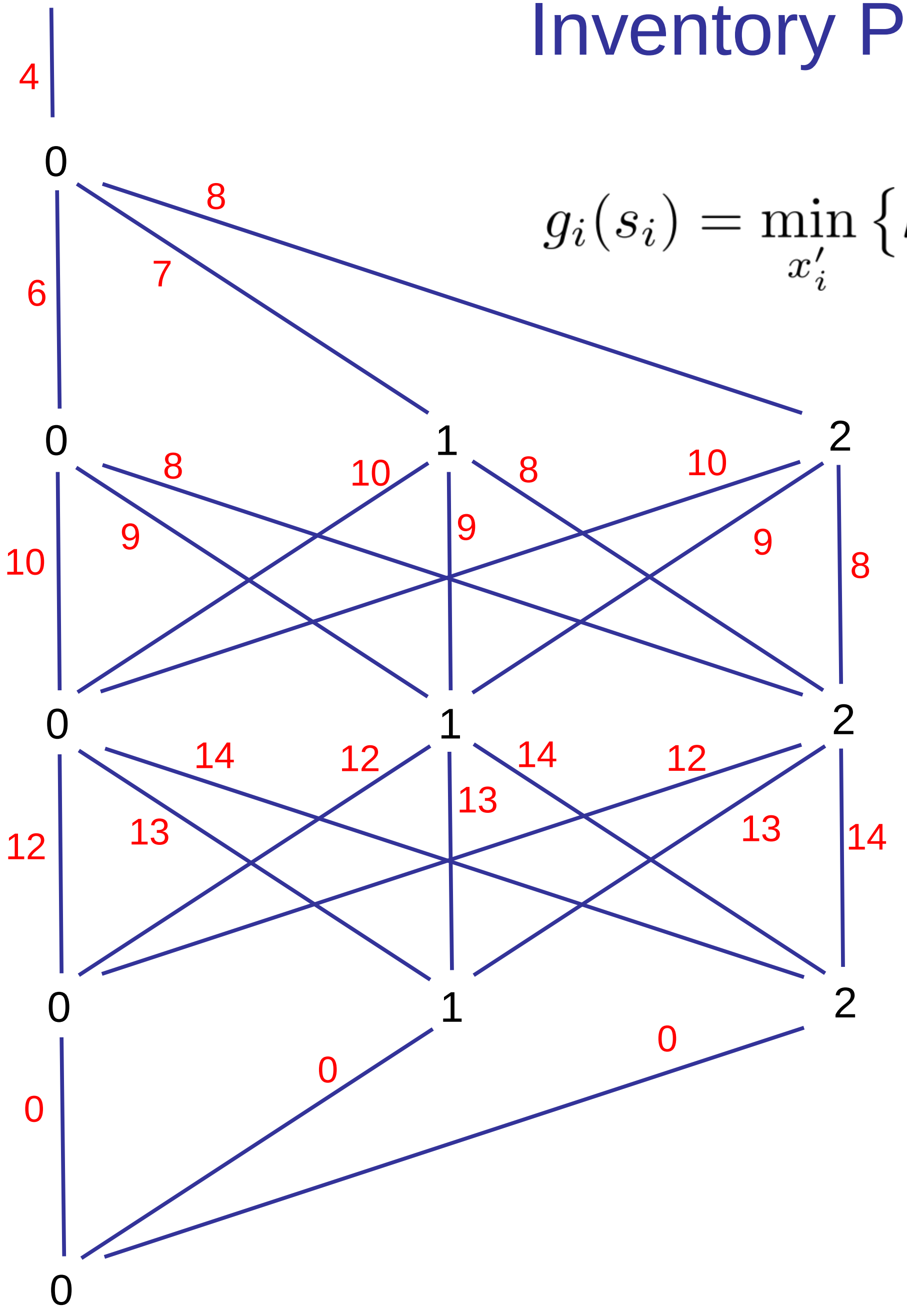
And all arcs into state s_i have the same cost

$$\bar{c}_i(s_{i+1}) = s_{i+1} h_{i+1} + c_i(d_i - s_{i+1} - m) + c_{i+1}(m - s_{i+1})$$

Inventory Problem

$$g_i(s_i) = \min_{x_i'} \left\{ h_i s_i + c_i(x_i' - s_i + d_i) + g_{i+1}(x_i') \right\}$$

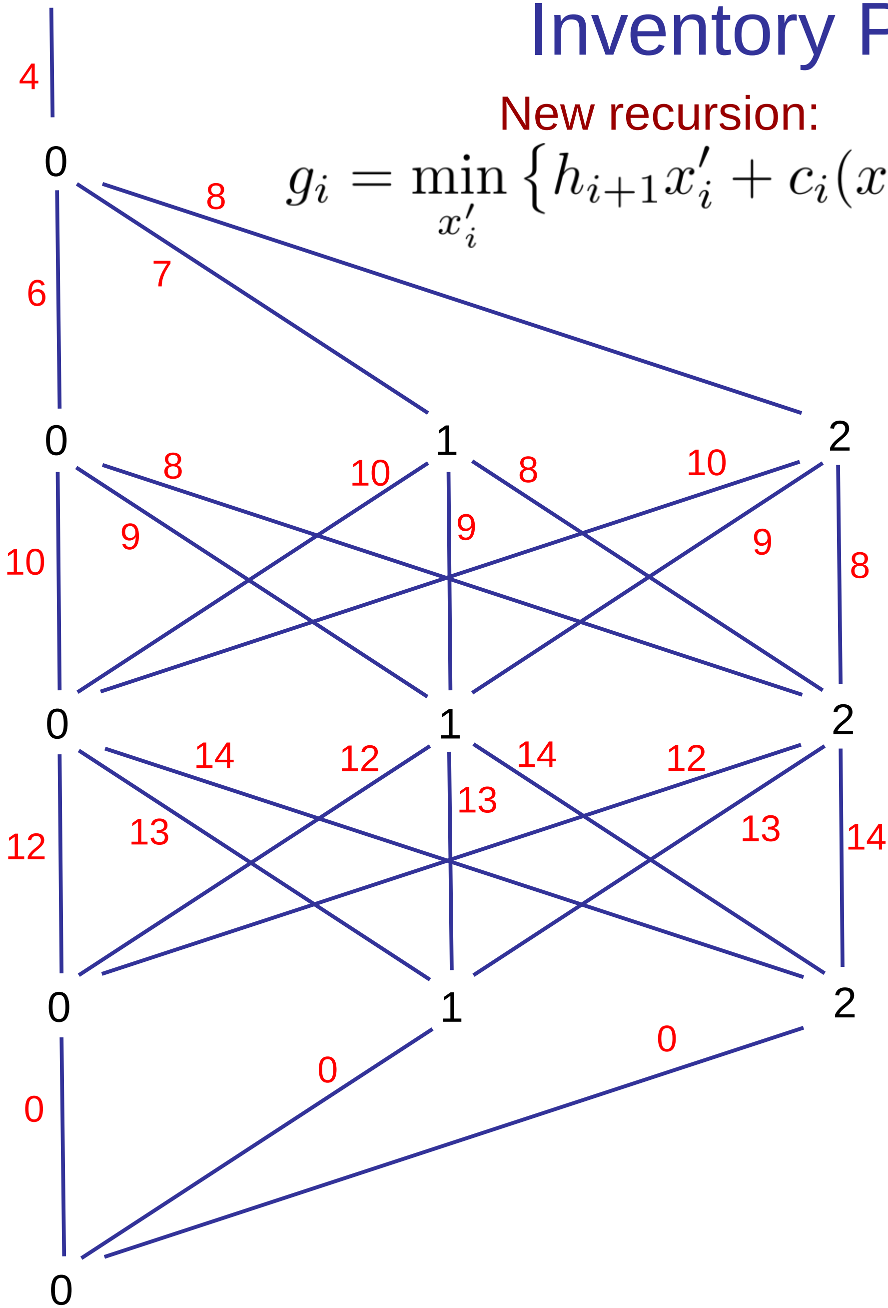
These are canonical costs with offset $\min_{s_{i+1}} \left\{ \bar{c}_i(s_{i+1}) \right\}$



Inventory Problem

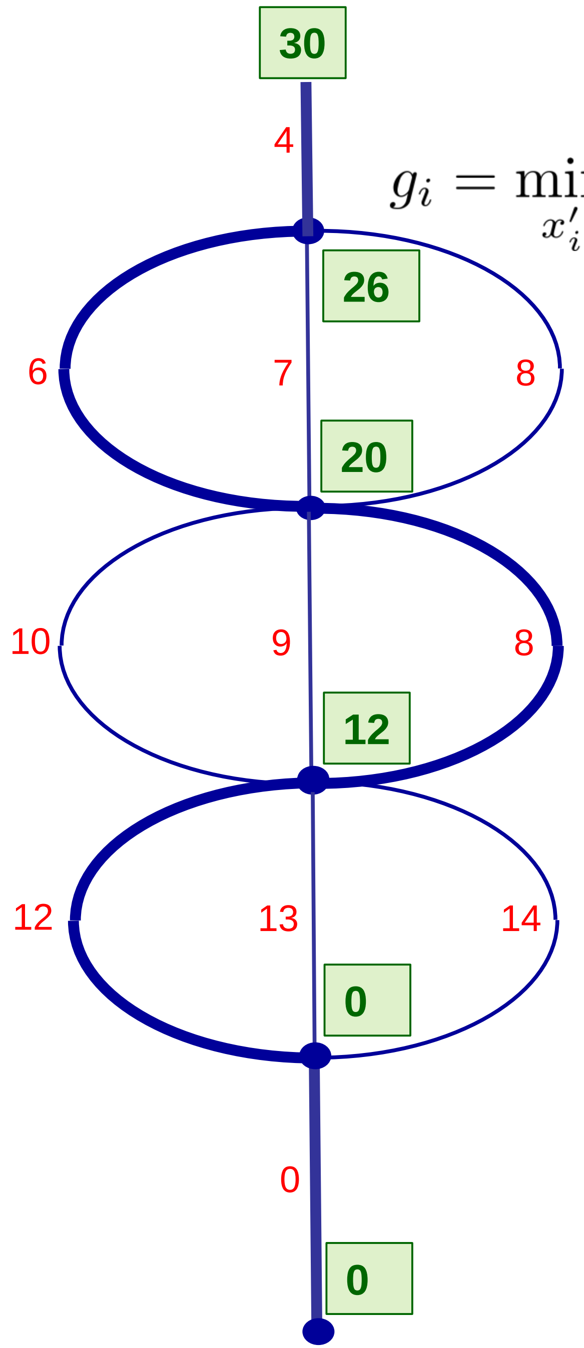
New recursion:

$$g_i = \min_{x_i'} \left\{ h_{i+1} x_i' + c_i(x_i' - m + d_i) + c_{i+1}(m - x_i') + g_{i+1} \right\}$$



These are canonical costs with offset $\min_{s_{i+1}} \left\{ \bar{c}_i(s_{i+1}) \right\}$

Inventory Problem



New recursion:

$$g_i = \min_{x_i'} \left\{ h_{i+1} x_i' + c_i(x_i' - m + d_i) + c_{i+1}(m - x_i') + g_{i+1} \right\}$$

Now there is only one state per period.

JH (2013)

Nonserial Decision Diagrams

- Analogous to **nonserial dynamic programming**, independently(?) rediscovered many times:
  - Nonserial DP (1972)
  - Constraint satisfaction (1981)
  - Data base queries (1983)
  - *k*-trees (1985)
  - Belief logics (1986)
  - Bucket elimination (1987)
  - Bayesian networks (1988)
  - Pseudoboolean optimization (1990)
  - Location analysis (1994)

Set Partitioning example

Find collection of sets that partition elements A, B, C, D

Sets

	1	2	3	4	5	6
A	●	●	●			
B		●		●		
C			●		●	●
D				●		●

Set Partitioning example

Find collection of sets that partition elements A, B, C, D

Sets

	1	2	3	4	5	6
A	●	●	●			
B		●		●		
C			●		●	●
D				●		●

For example…

Set Partitioning example

Find collection of sets that partition elements A, B, C, D

Sets

	1	2	3	4	5	6
A	•	•	•			
B		•		•		
C			•		•	•
D				•		•

Or…

Set Partitioning example

Find collection of sets that partition elements A, B, C, D

Sets

	1	2	3	4	5	6
A	•	•	•			
B		•		•		
C			•		•	•
D				•		•

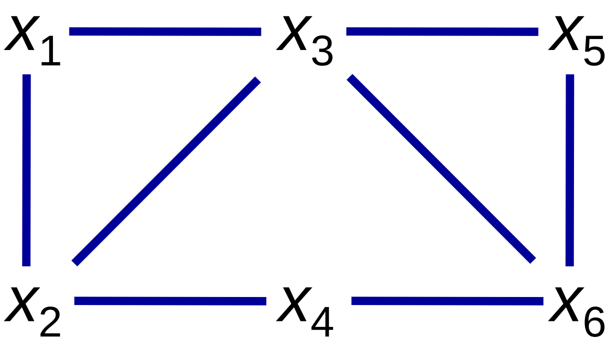
0-1 formulation

$$
\begin{aligned}
x_1 + x_2 + x_3 \phantom{{}+x_4+x_5+x_6} &= 1 \\
x_2 \phantom{{}+x_3} + x_4 \phantom{{}+x_5+x_6} &= 1 \\
x_3 \phantom{{}+x_4} + x_5 + x_6 &= 1 \\
x_4 \phantom{{}+x_5} + x_6 &= 1
\end{aligned}
$$

$x_j = 1 \quad \Rightarrow \quad$ set j selected

Set Partitioning example

Dependency graph



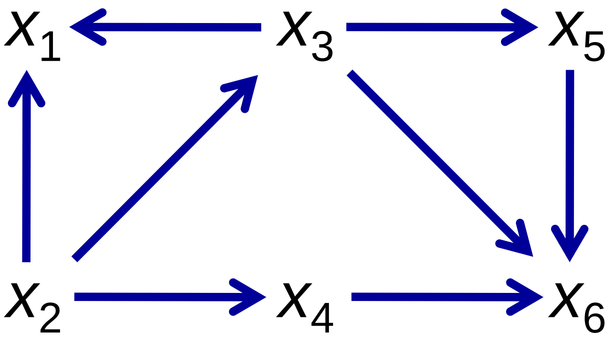
0-1 formulation

$$
\begin{aligned}
x_1 + x_2 + x_3 \quad\quad\quad\quad\quad\quad &= 1 \\
x_2 \quad\quad + x_4 \quad\quad\quad\quad &= 1 \\
x_3 \quad\quad + x_5 + x_6 &= 1 \\
x_4 \quad\quad + x_6 &= 1
\end{aligned}
$$

$x_j = 1 \quad \Rightarrow \quad$ set j selected

Set Partitioning example

Dependency graph



Enumeration order

x_2

x_3

x_4

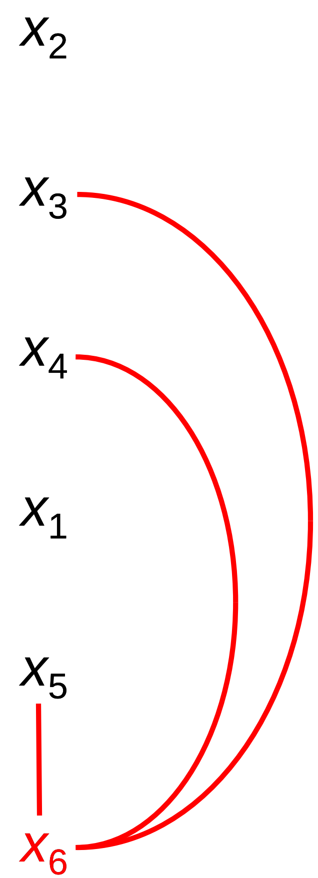
x_1

x_5

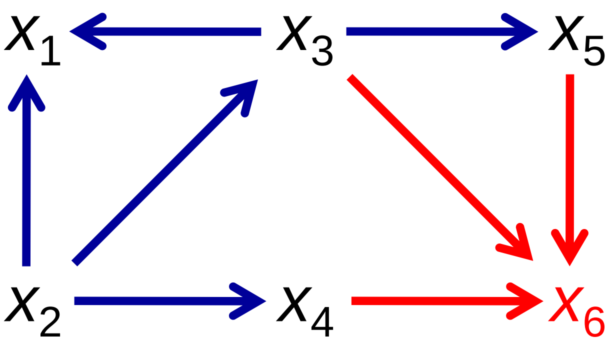
x_6

Set Partitioning example

Enumeration order

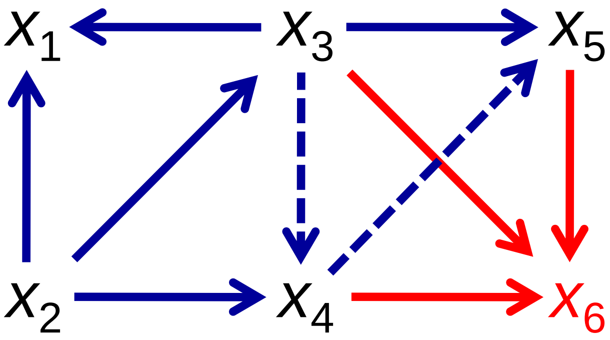


Dependency graph

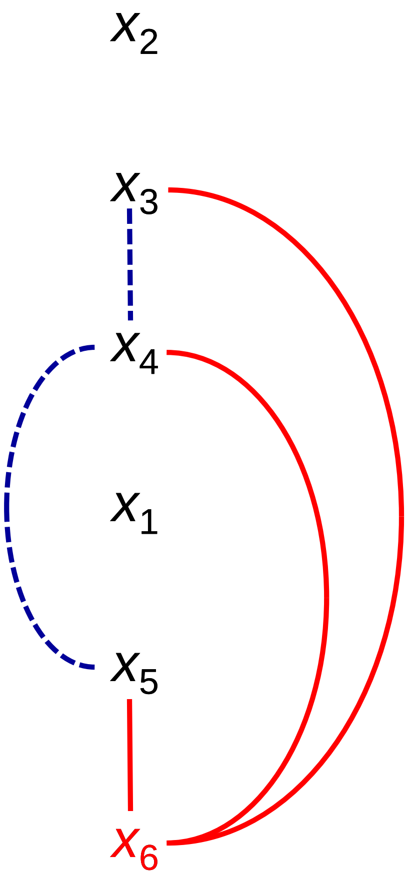


Set Partitioning example

Dependency graph

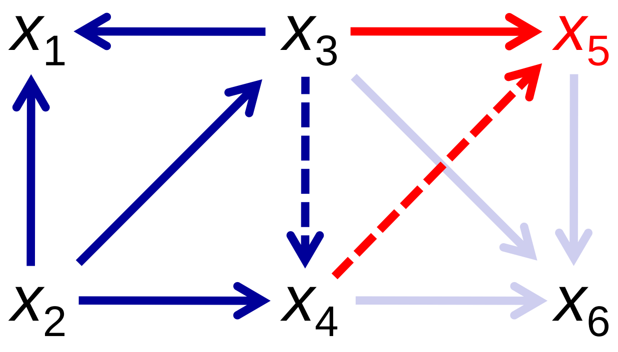


Enumeration order

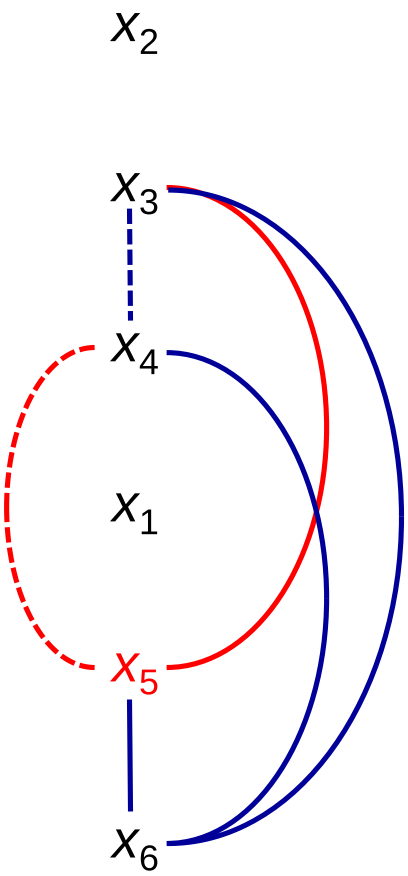


Set Partitioning example

Dependency graph

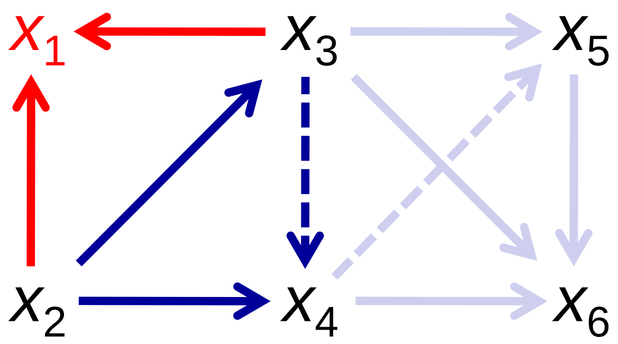


Enumeration order

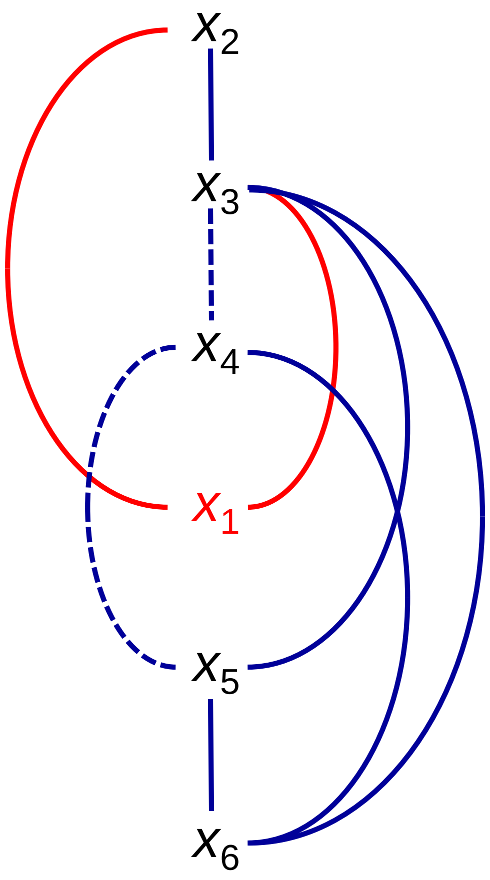


Set Partitioning example

Dependency graph

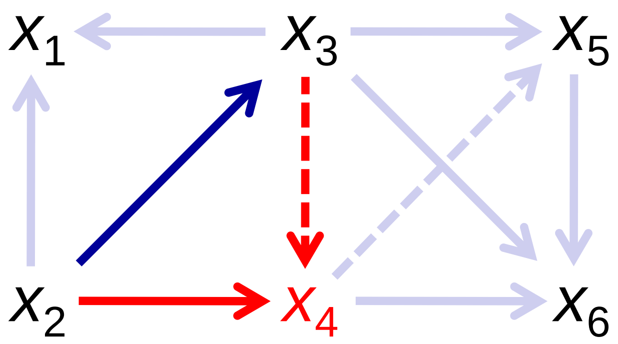


Enumeration order

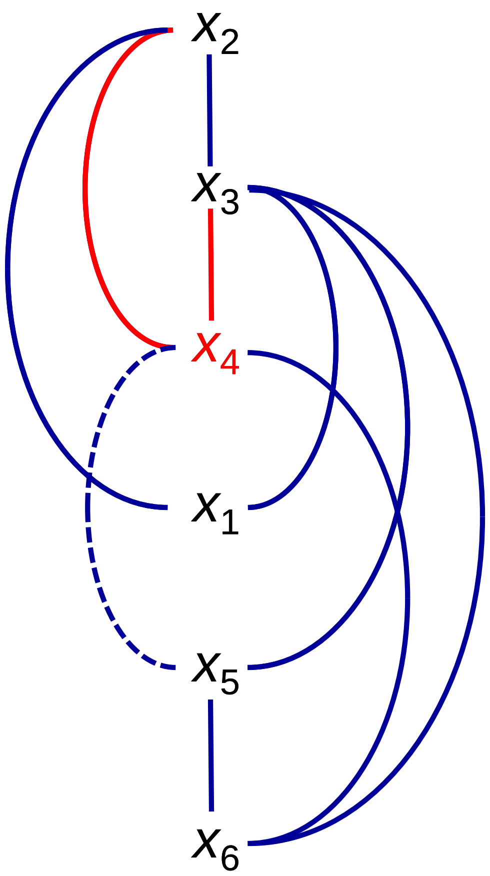


Set Partitioning example

Dependency graph

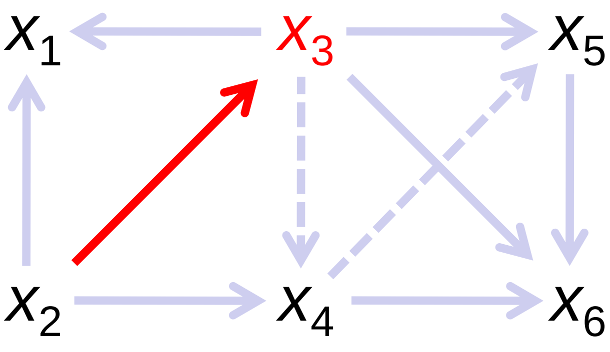


Enumeration order

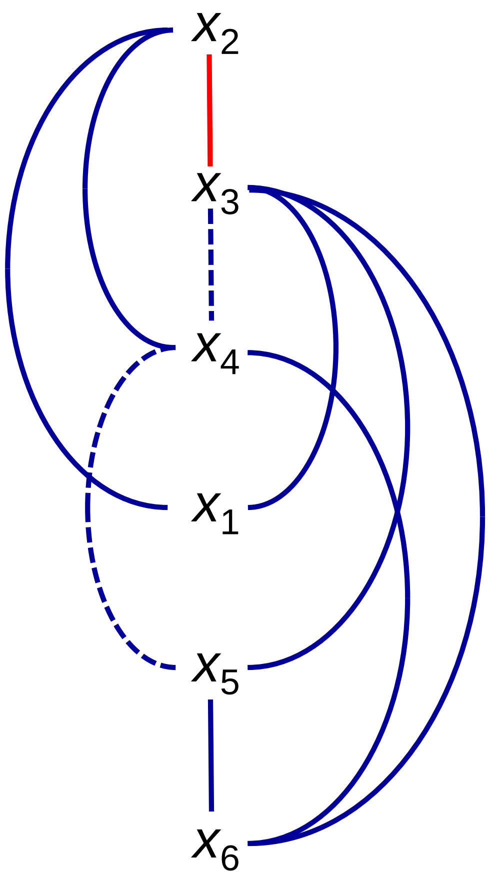


Set Partitioning example

Dependency graph

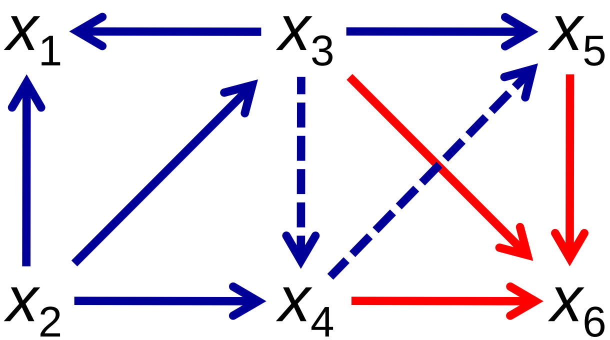


Enumeration order



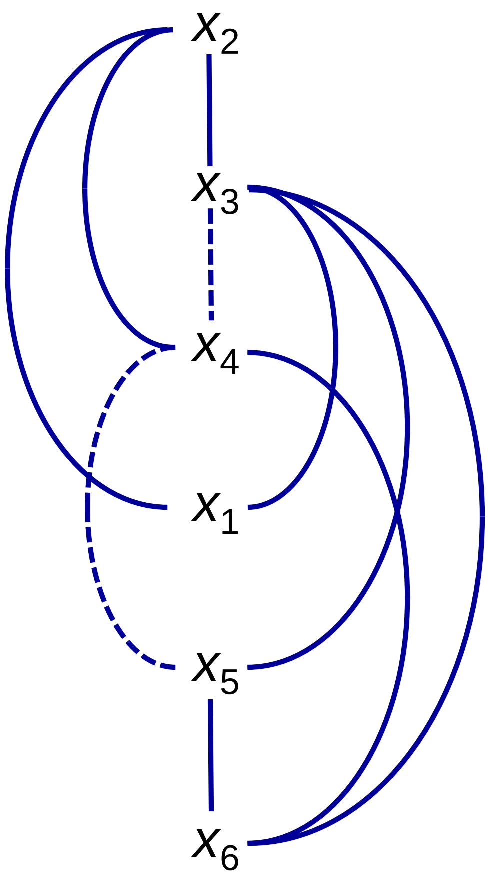
Set Partitioning example

Dependency graph



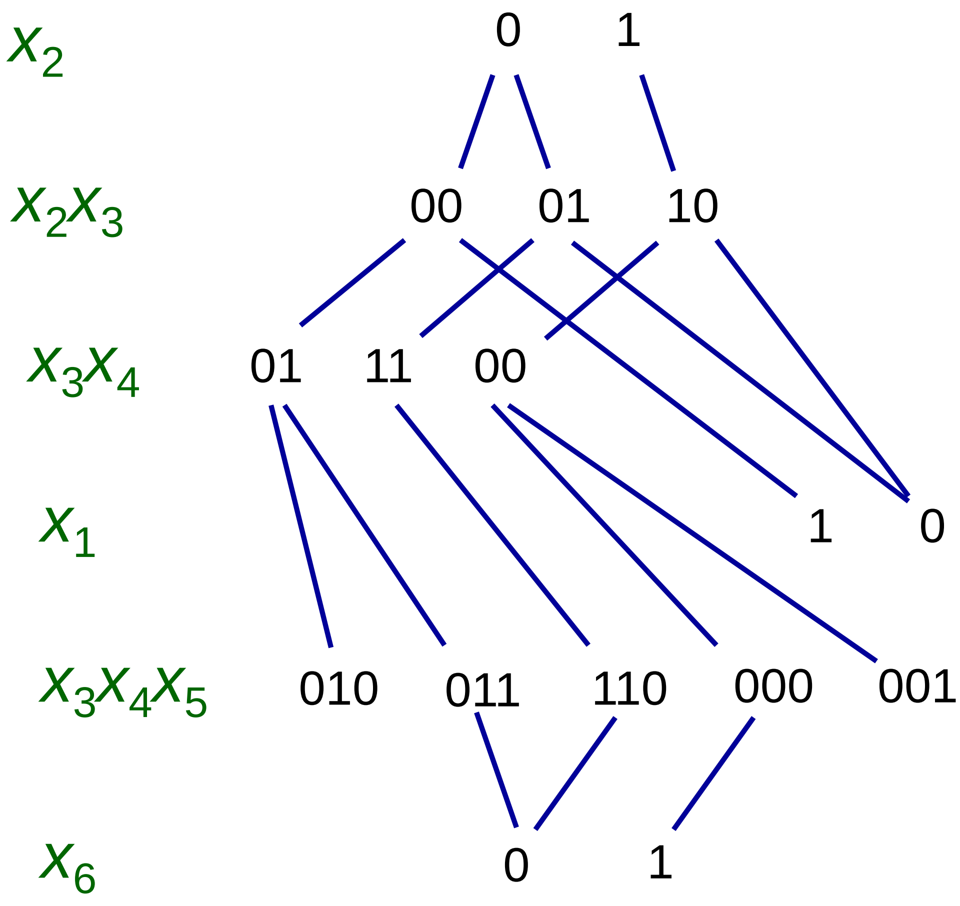
Induced width = 3
(max in-degree)

Enumeration order

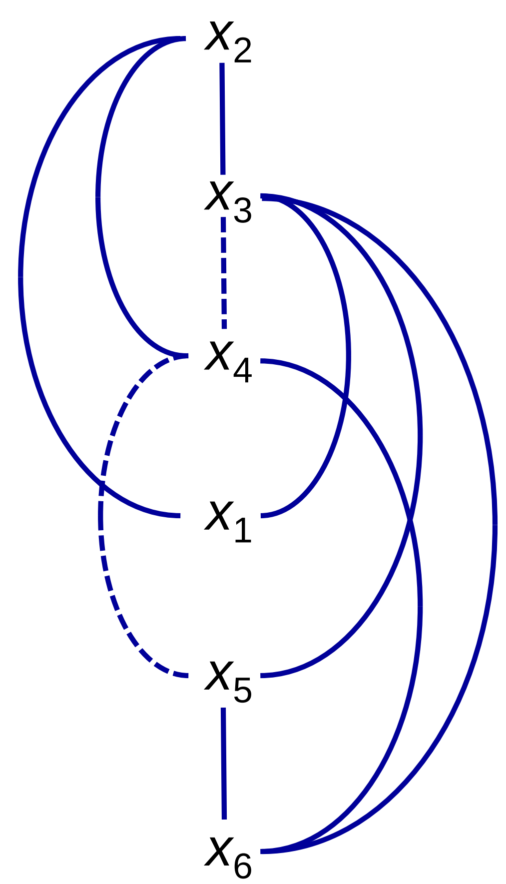


Set Partitioning example

Solution by nonserial DP

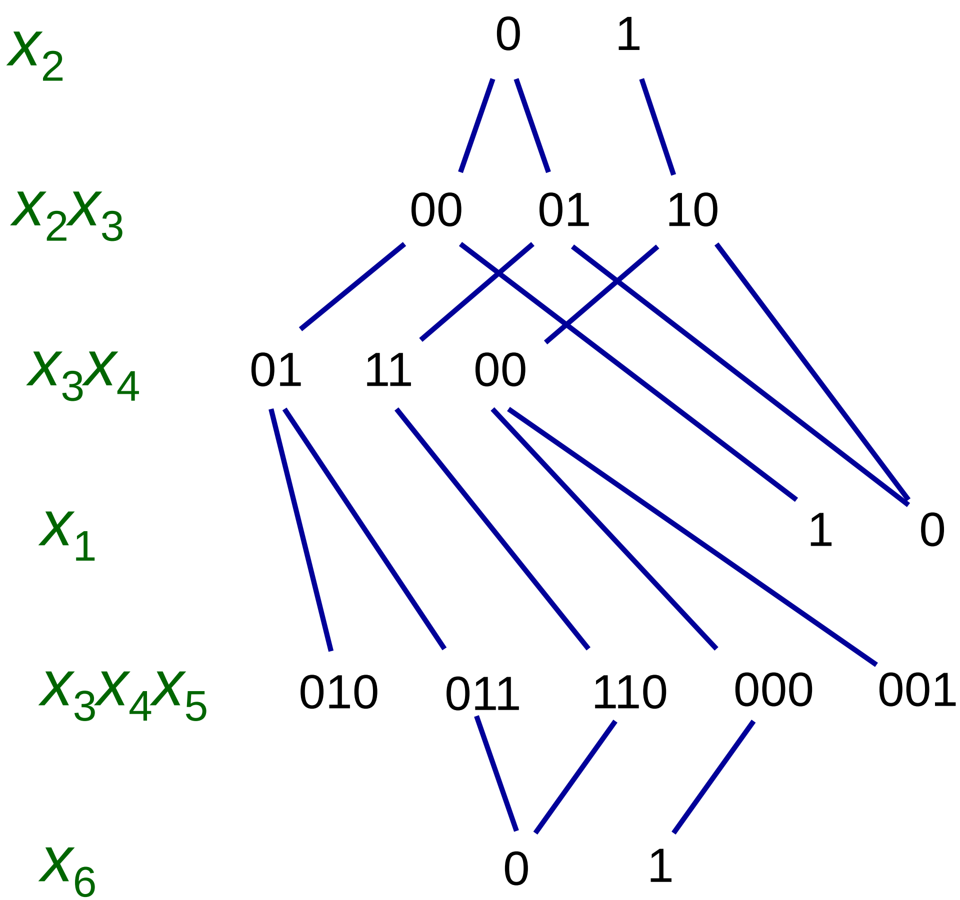


Enumeration order



Set Partitioning example

Solution by nonserial DP

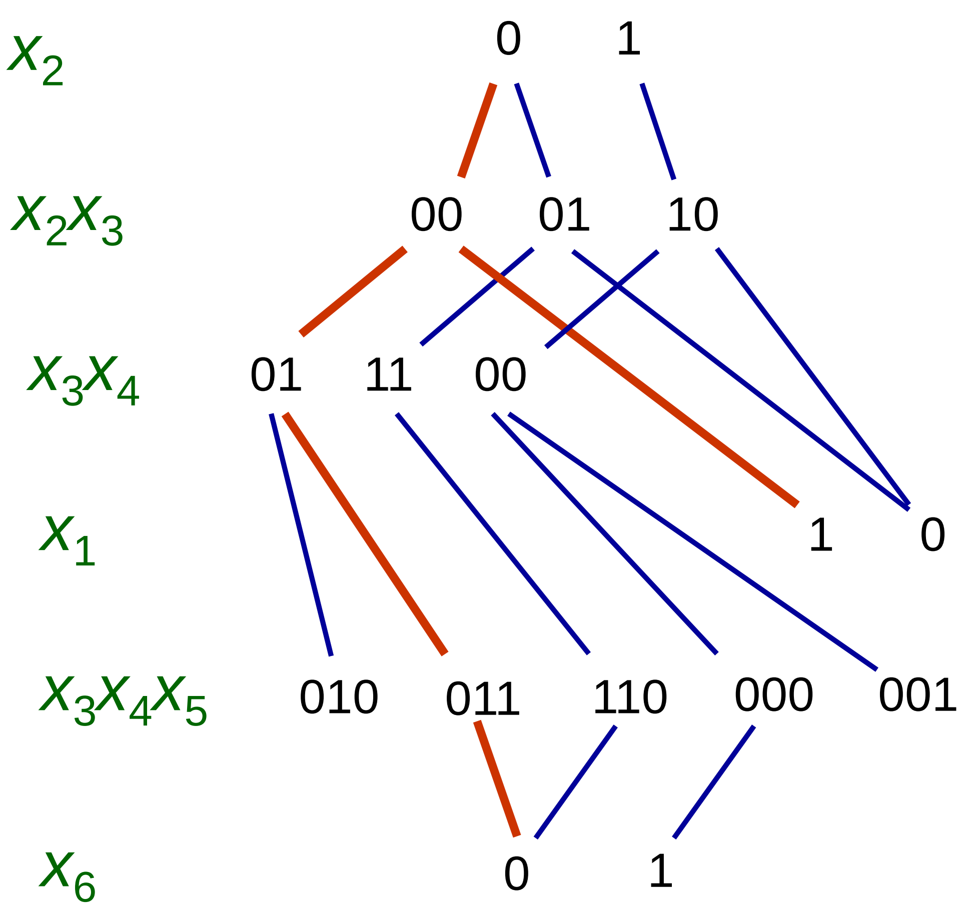


Sets

	1	**2**	**3**	**4**	**5**	**6**
A	●	●	●			
B		●		●		
C			●		●	●
D				●		●

Set Partitioning example

Feasible solution

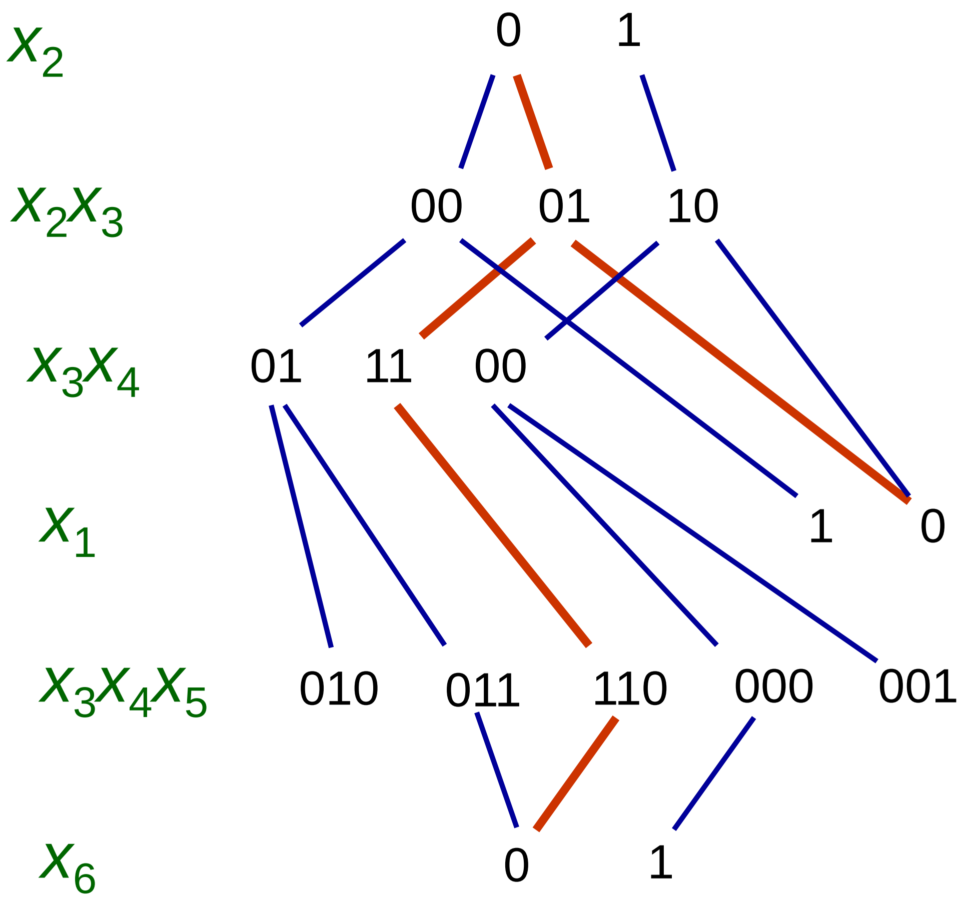


Sets

	1	**2**	**3**	**4**	**5**	**6**
A	●	●	●			
B		●		●		
C			●		●	●
D				●		●
	1	0	0	1	1	0

Set Partitioning example

Feasible solution



Sets

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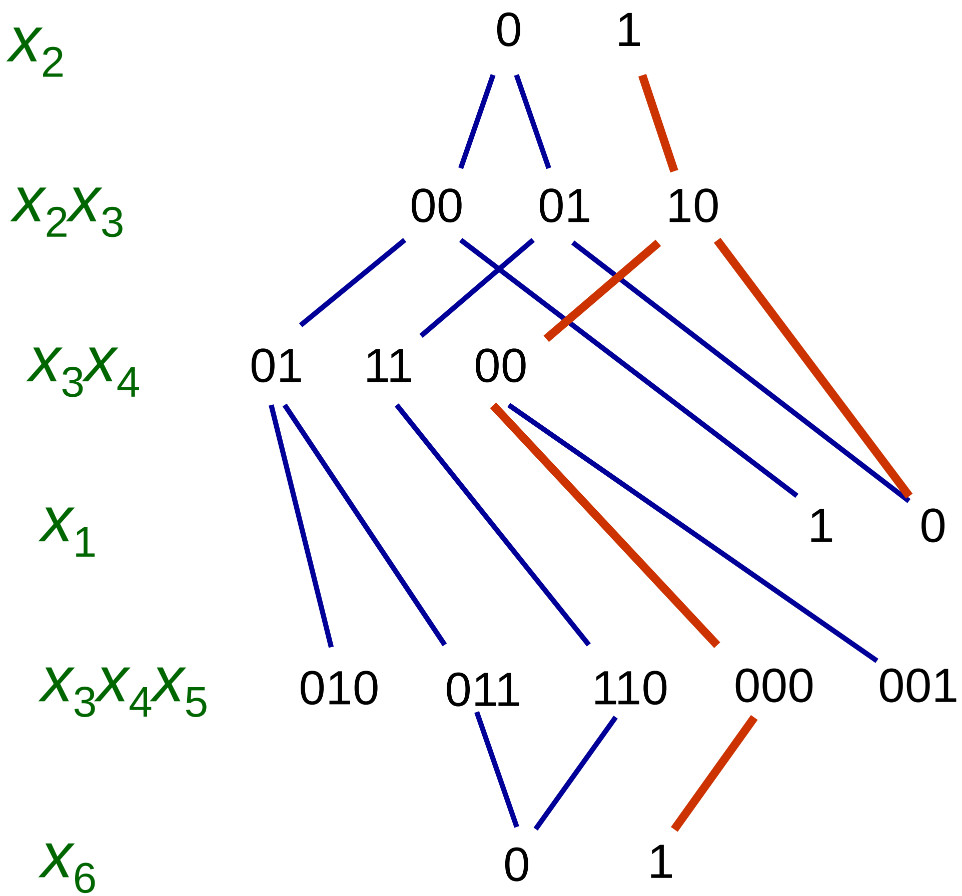
A	●	●	●			
B		●		●		
C			●		●	●
D				●		●
	1	0	0	1	1	0
	0	0	1	1	0	0

Set Partitioning example

Feasible solution



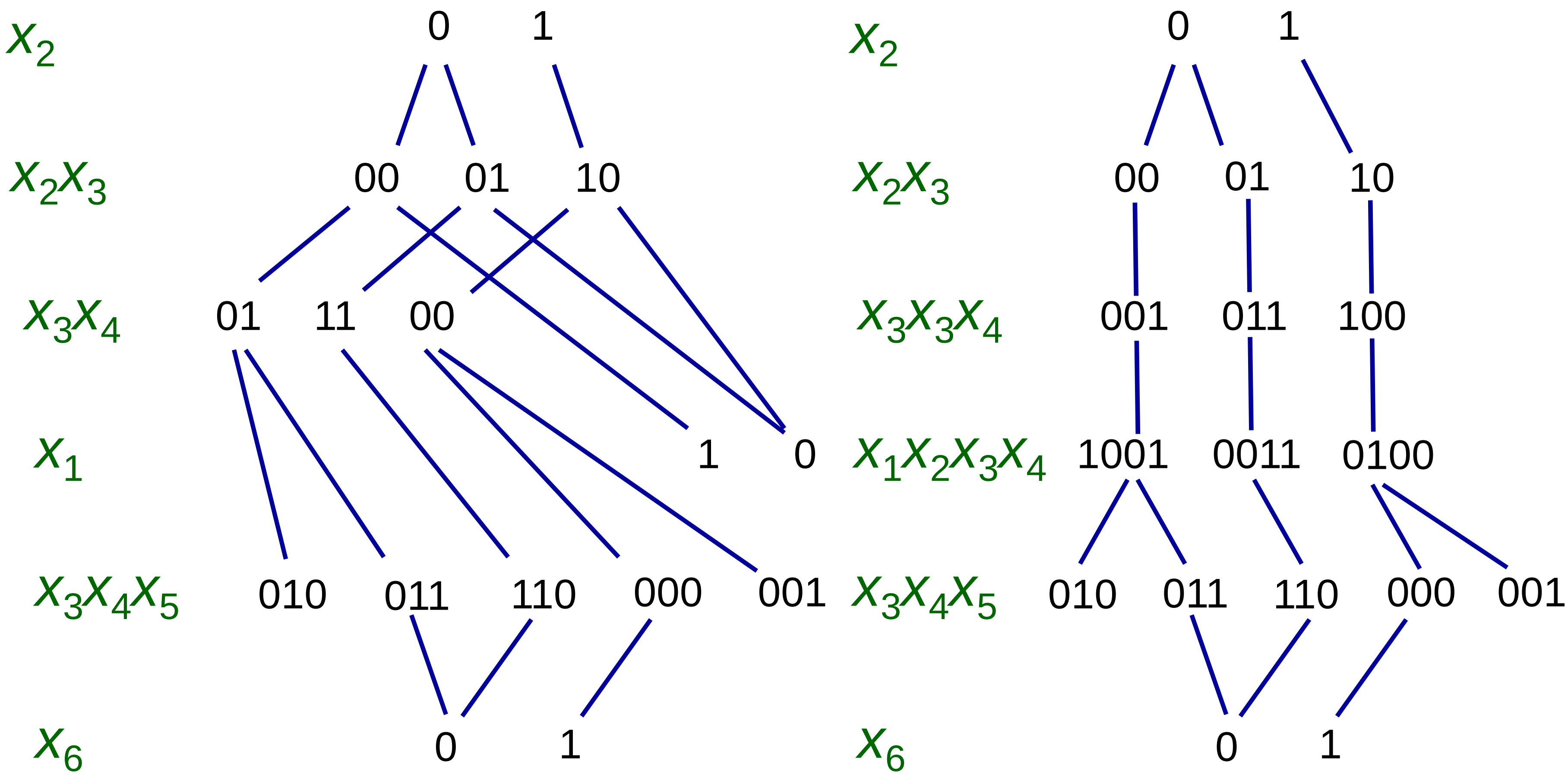
Sets

	1	**2**	**3**	**4**	**5**	**6**
A	●	●	●			
B		●		●		
C			●		●	●
D				●		●
	1	0	0	1	1	0
	0	0	1	1	0	0
	0	1	0	0	0	1

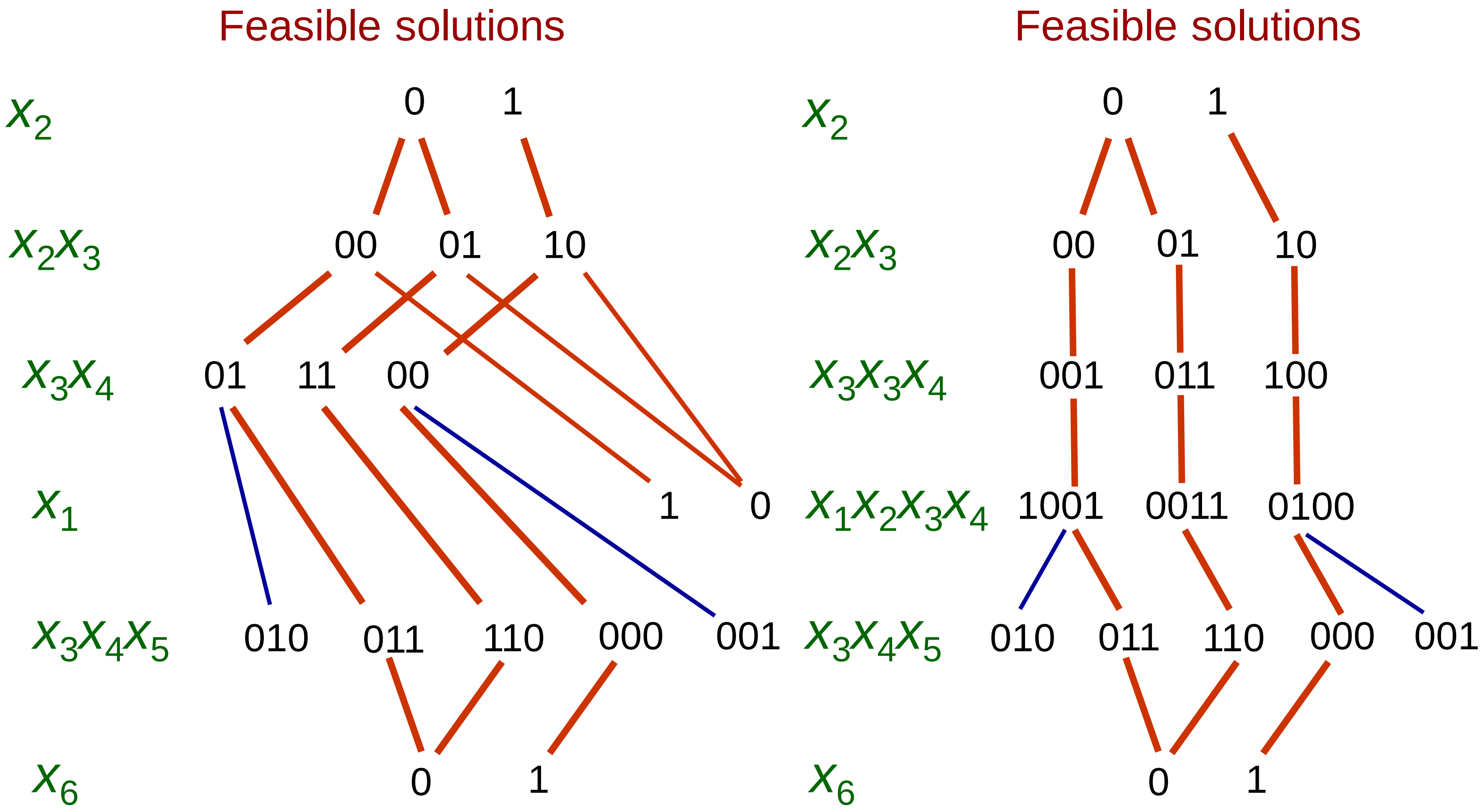
Set Partitioning example

Solution by nonserial DP

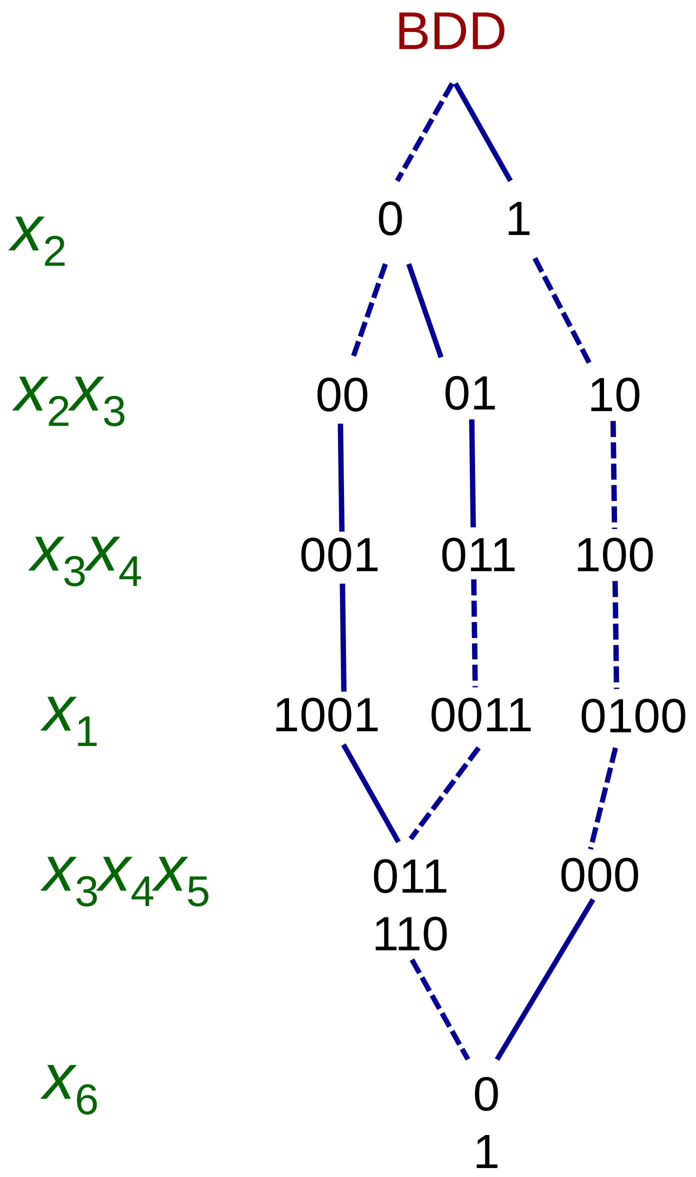
Serialized DP

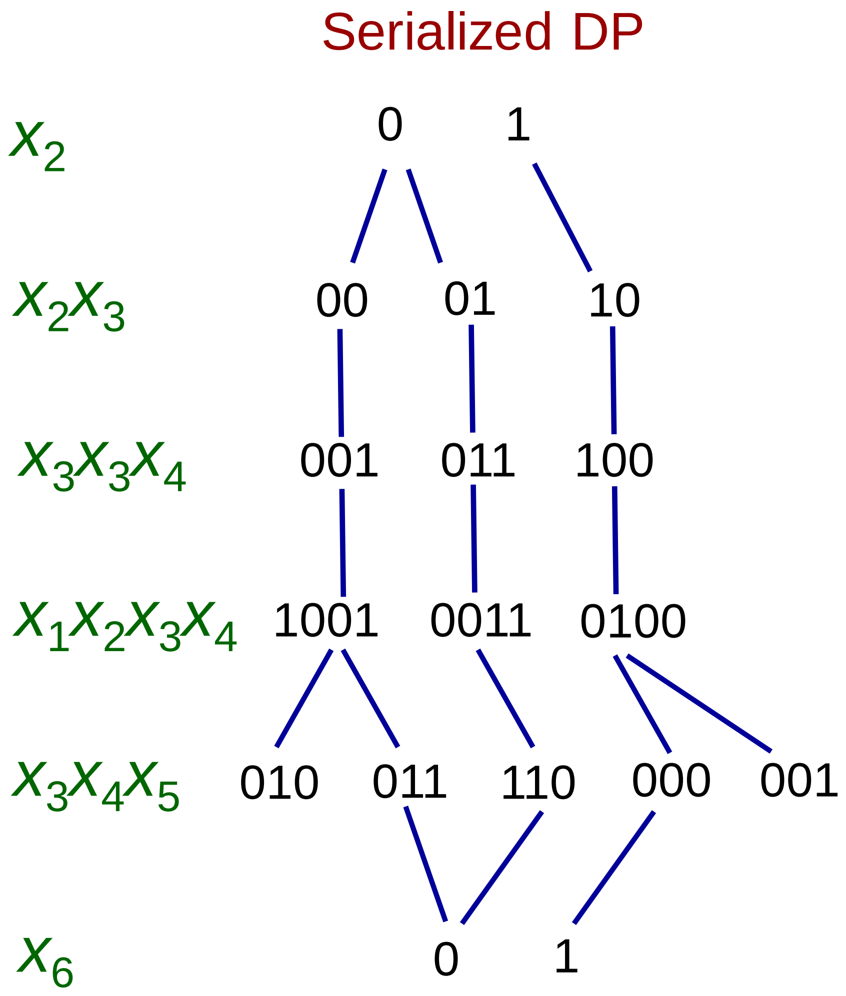


Set Partitioning example

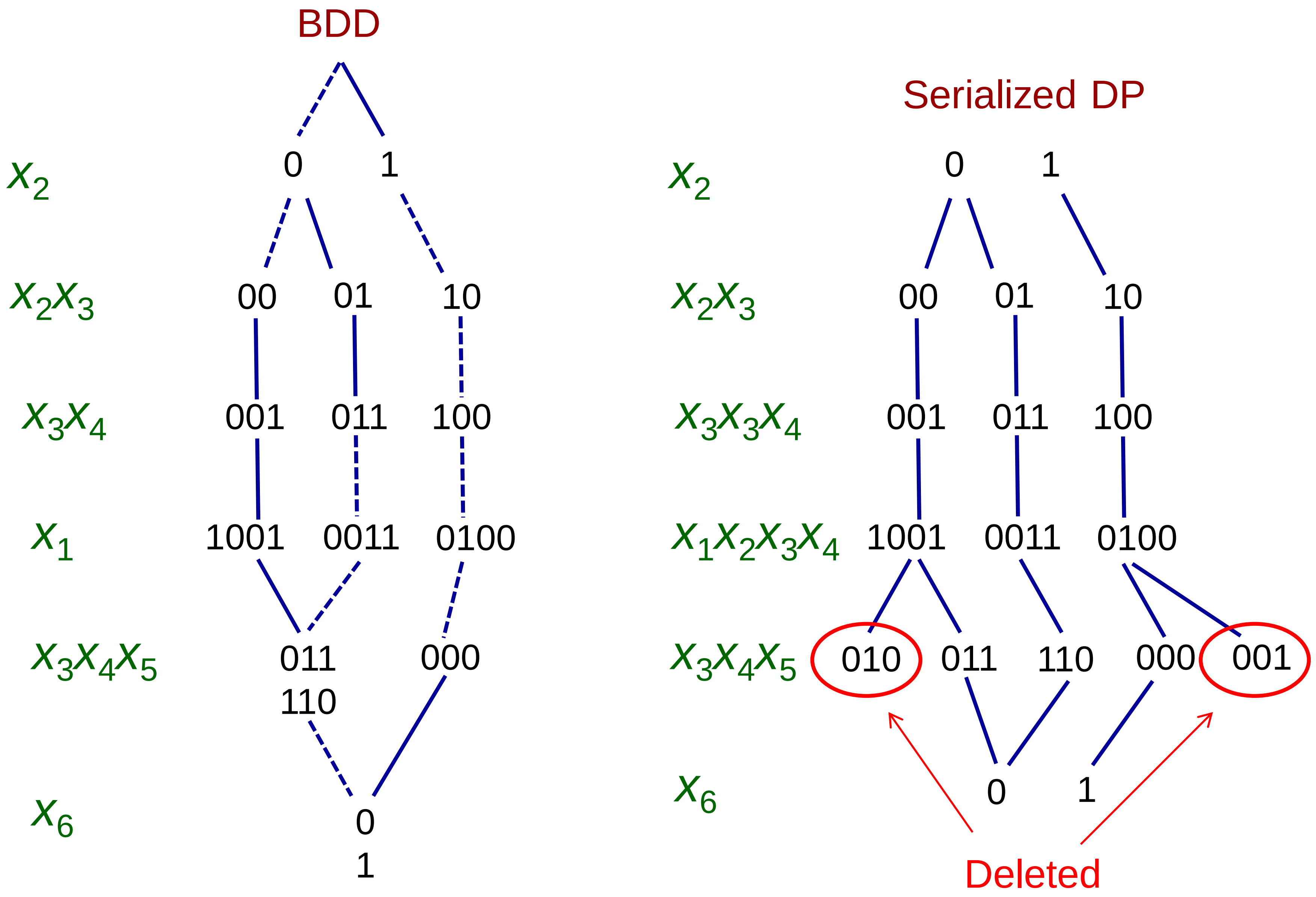


BDD vs. DP Solution

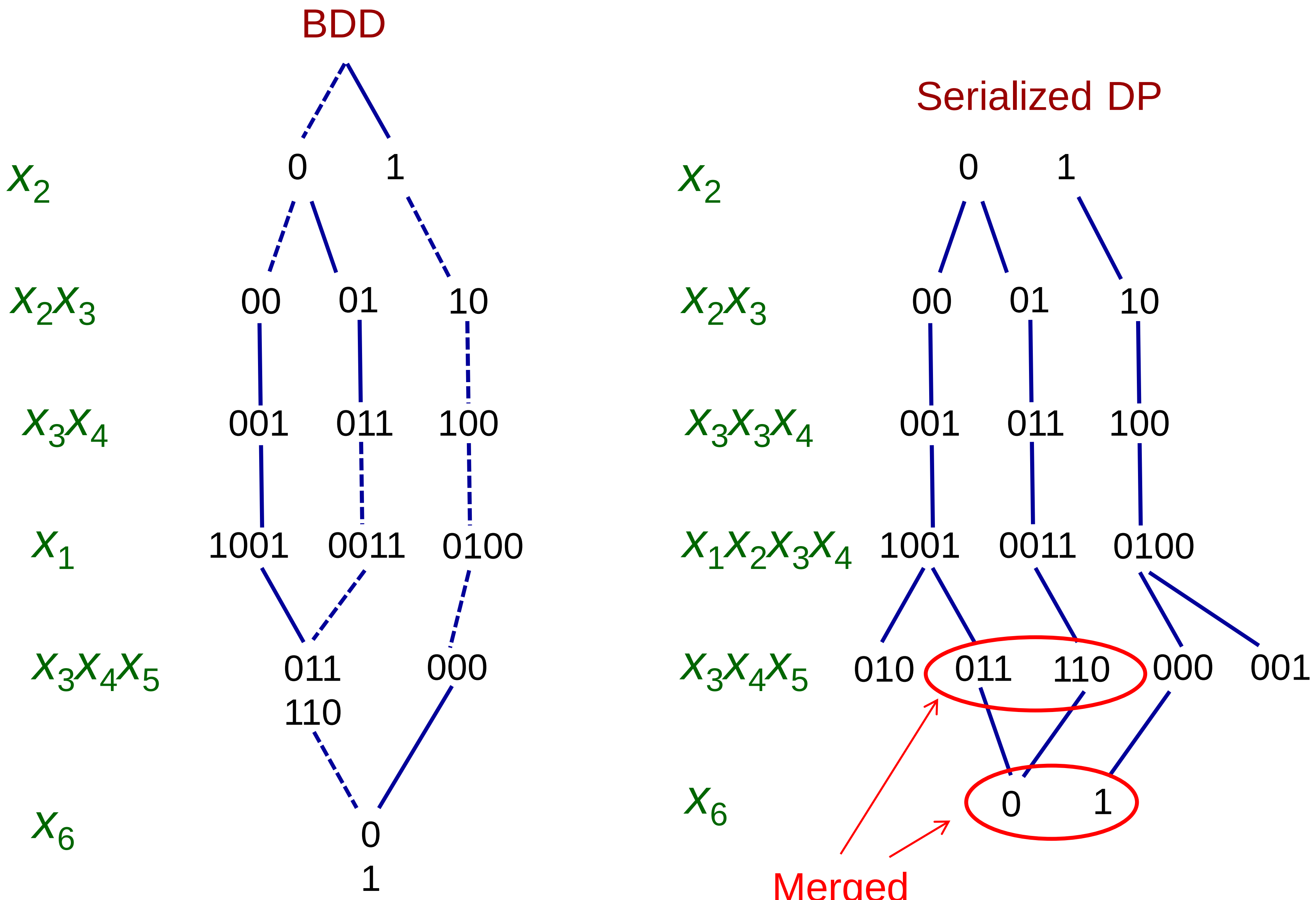




BDD vs. DP Solution

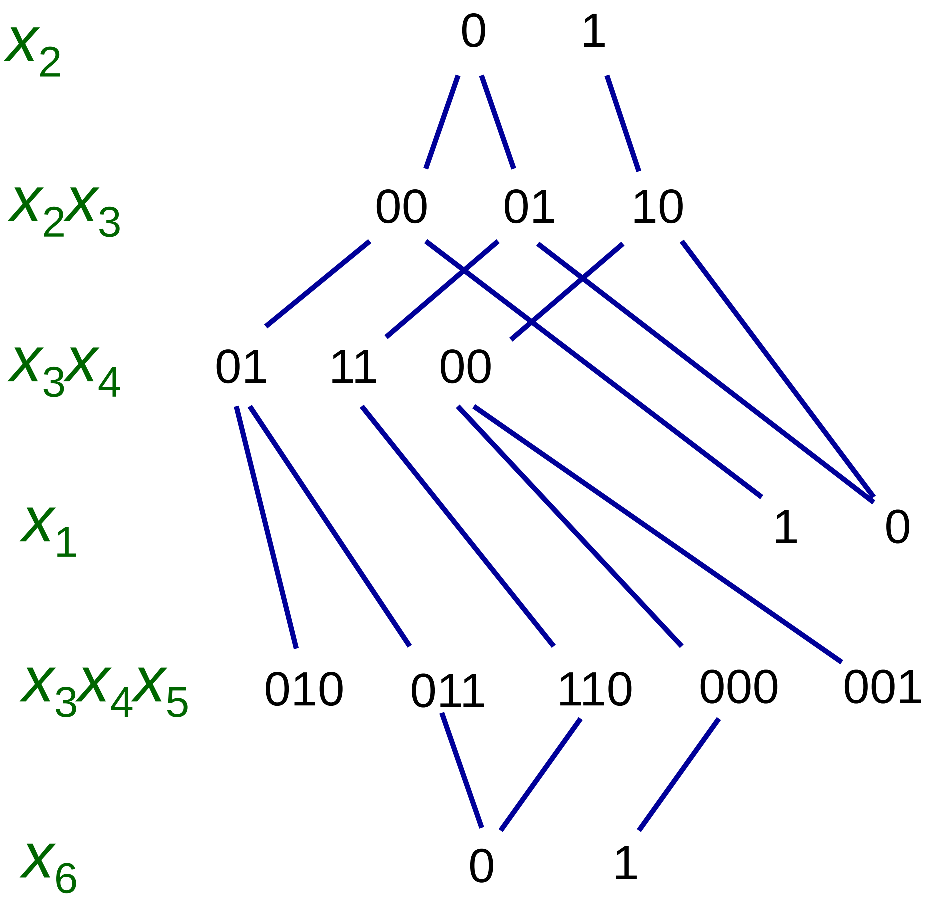


BDD vs. DP Solution

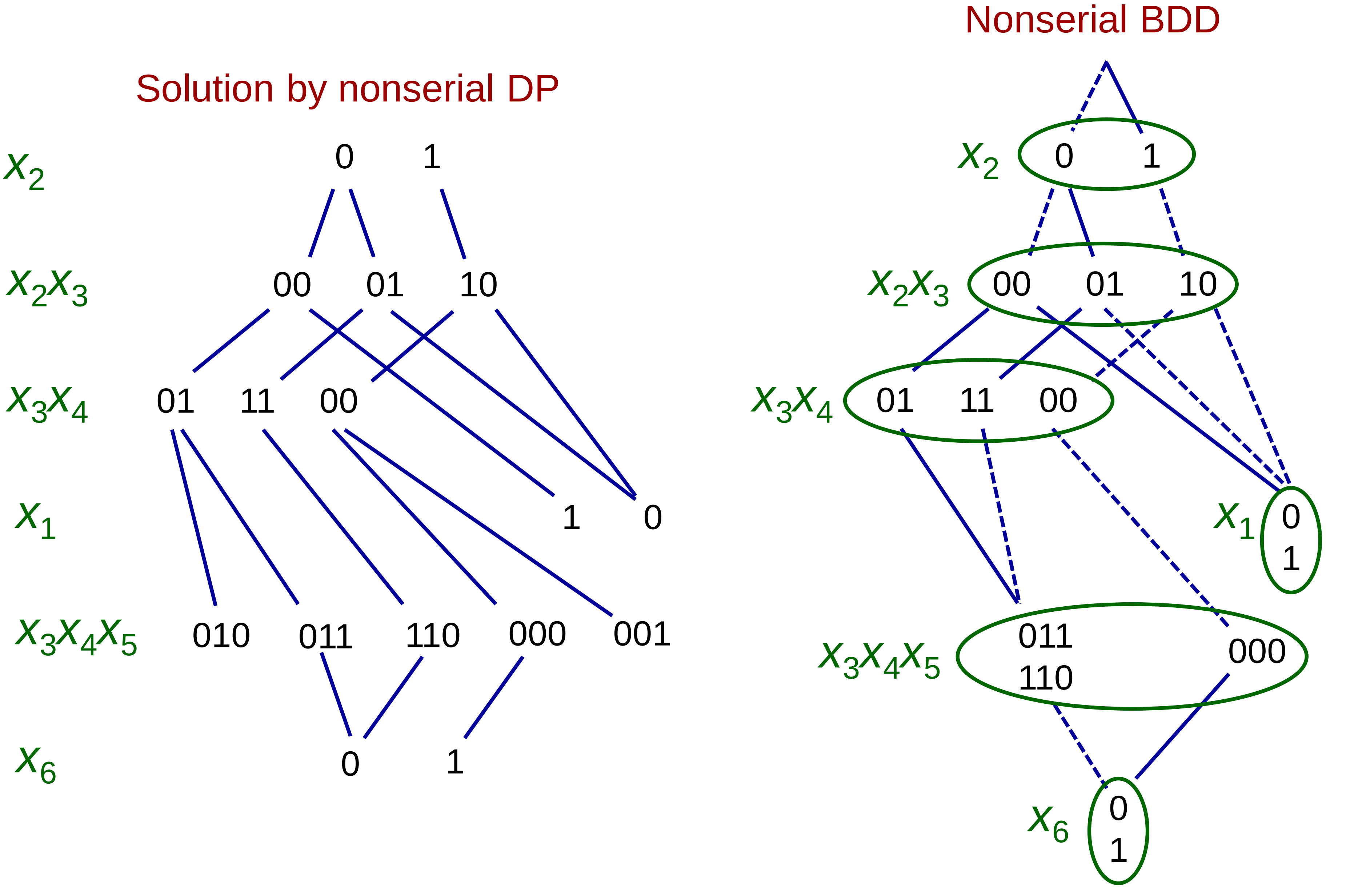


Set Partitioning example

Solution by nonserial DP



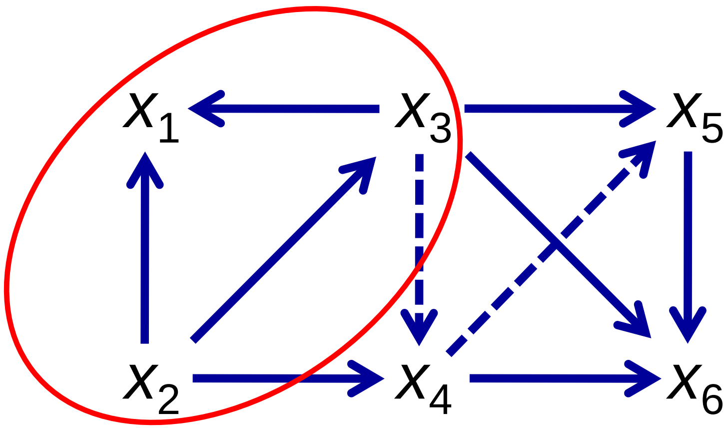
Set Partitioning example



Constructing the Join Tree

$x_2 \ x_3 \ x_4 \ x_1 \ x_5 \ x_6$

Clique in the dependency graph



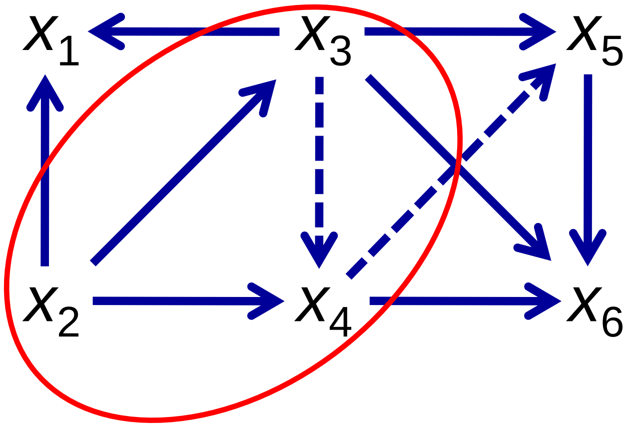
$x_1 x_2 x_3$

Constructing the Join Tree

$x_2\ x_3\ x_4\ x_1\ x_5\ x_6$

Clique in the
dependency graph

$x_1x_2x_3$

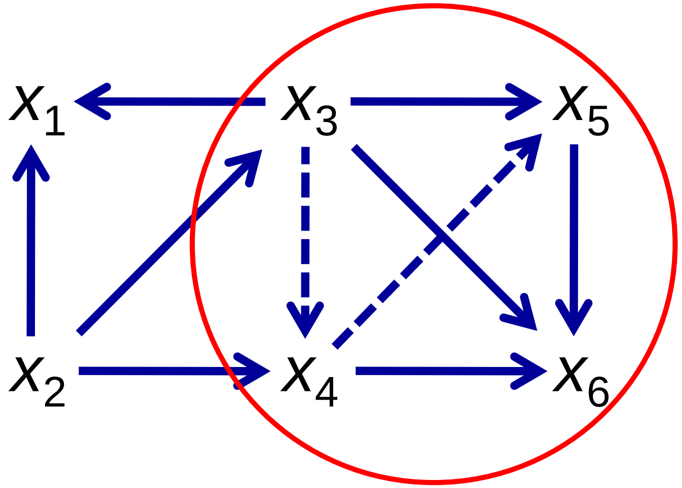


$x_2x_3x_4$

Constructing the Join Tree

$x_2 \; x_3 \; x_4 \; x_1 \; x_5 \; x_6$

Clique in the dependency graph



$x_1x_2x_3$

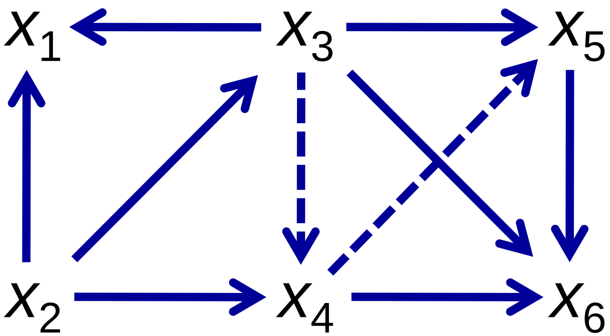
$x_2x_3x_4$

$x_3x_4x_5x_6$

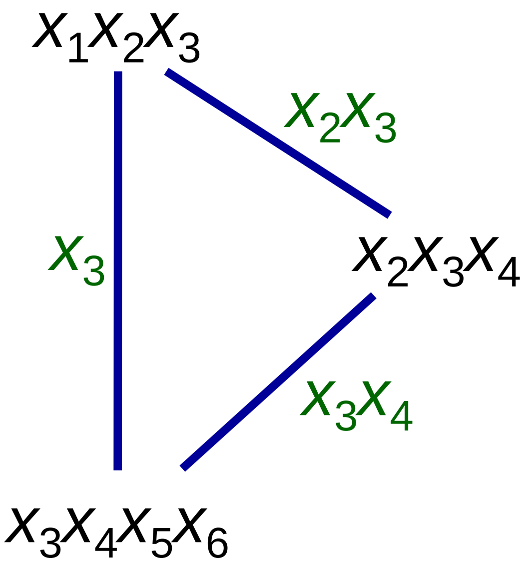
Constructing the Join Tree

$x_2\ x_3\ x_4\ x_1\ x_5\ x_6$

Dependency graph



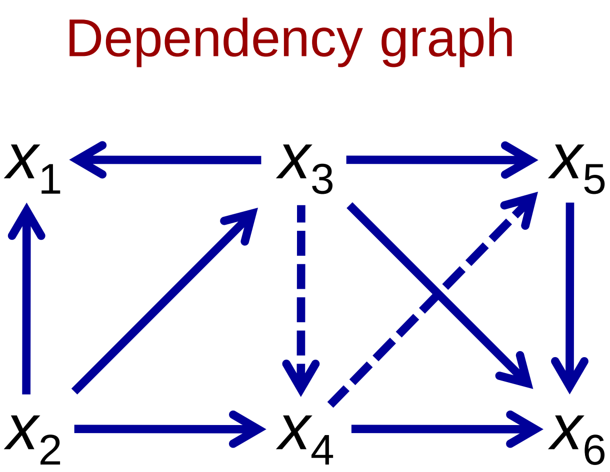
Join graph

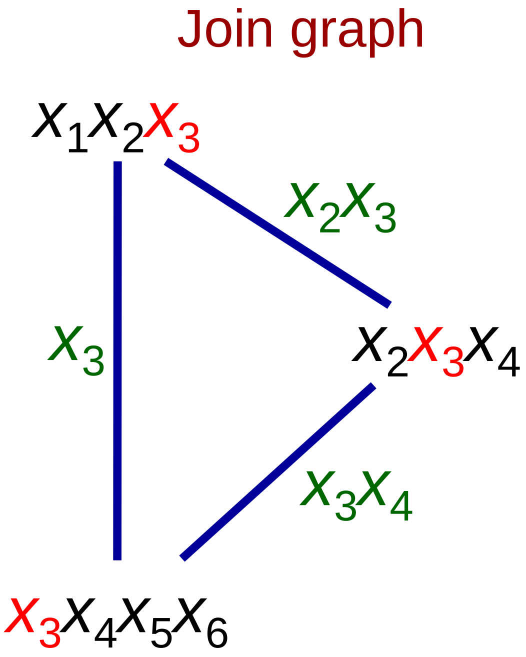


Connect nodes with common variables

Constructing the Join Tree

$x_2\ x_3\ x_4\ x_1\ x_5\ x_6$

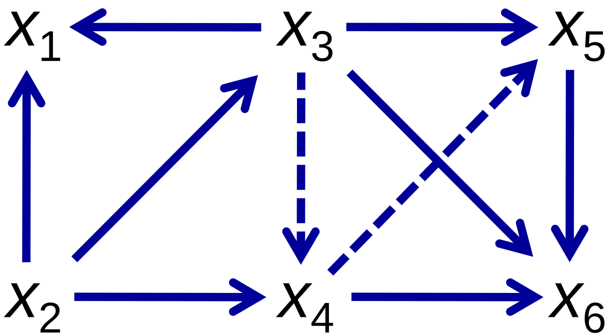




x_j occurs along every path connecting x_i with x_j

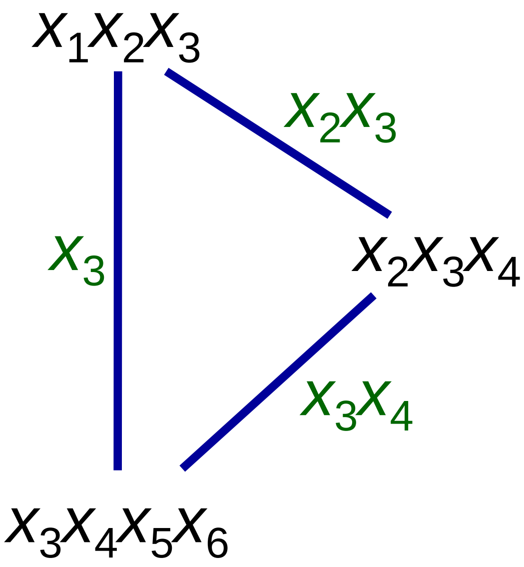
Constructing the Join Tree

$x_2\ x_3\ x_4\ x_1\ x_5\ x_6$



Dependency graph

Join graph



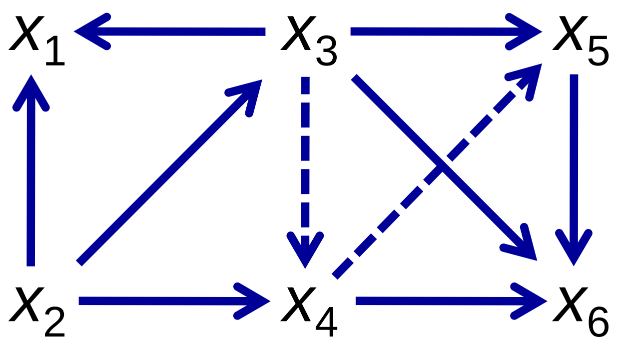
This can be viewed as the **constraint dual**

Binary constraints equate common variables in subproblems

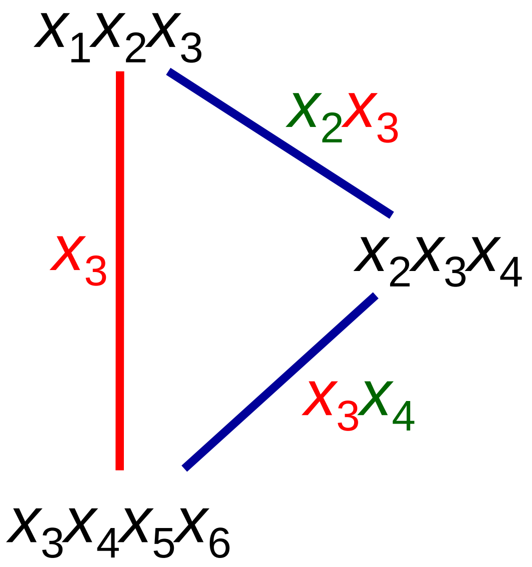
Constructing the Join Tree

$x_2 \; x_3 \; x_4 \; x_1 \; x_5 \; x_6$

Dependency graph



Join graph

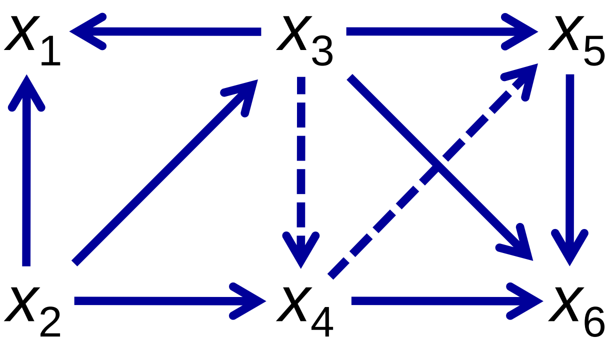


Some edges may be redundant
when equating variables

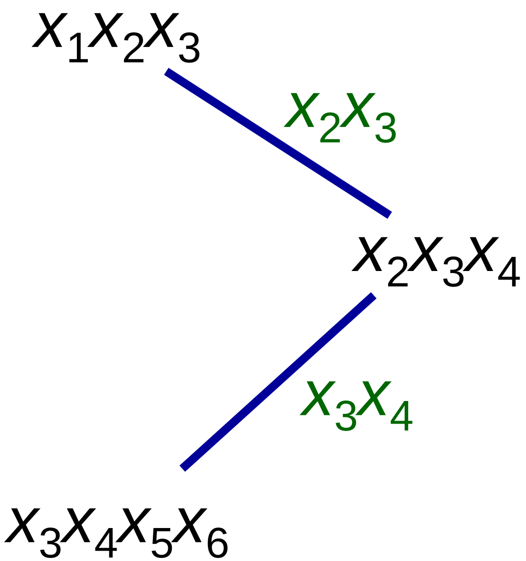
Constructing the Join Tree

$x_2 \; x_3 \; x_4 \; x_1 \; x_5 \; x_6$

Dependency graph



Join tree

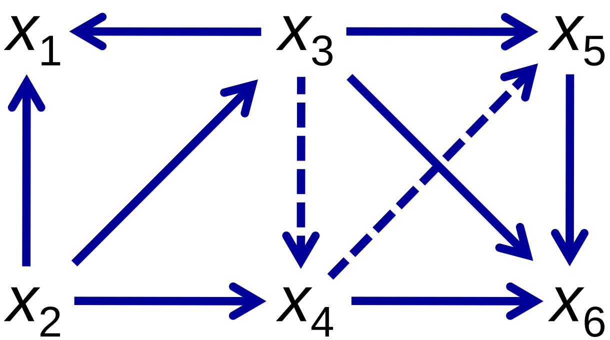


Removing redundant edges
yields **join tree**

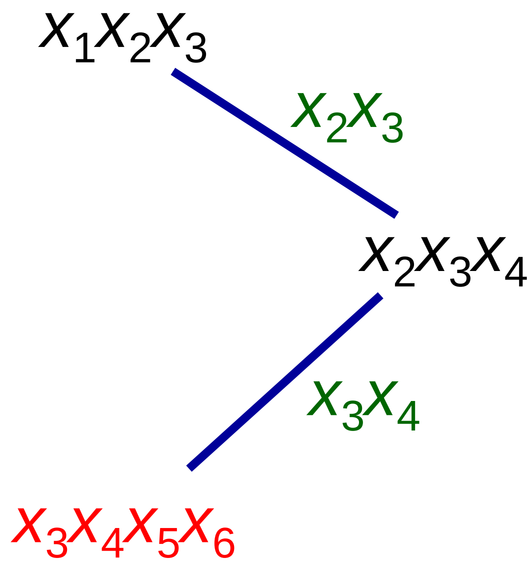
Constructing the Join Tree

$X_2 \; X_3 \; X_4 \; X_1 \; X_5 \; X_6$

Dependency graph



Join tree

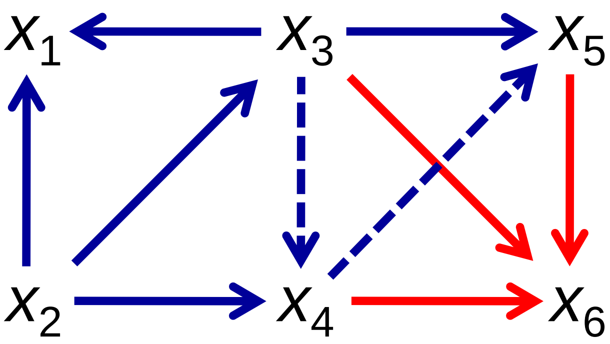


Max node cardinality is
tree width + 1 = **3** + 1

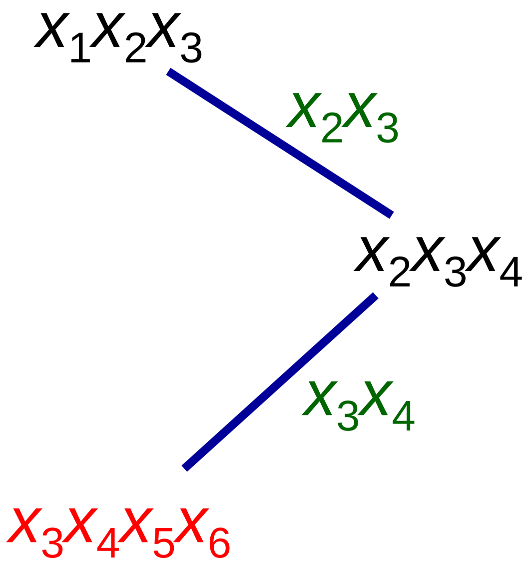
Constructing the Join Tree

$x_2 \ x_3 \ x_4 \ x_1 \ x_5 \ x_6$

Dependency graph



Join tree



Induced width = tree width = 3

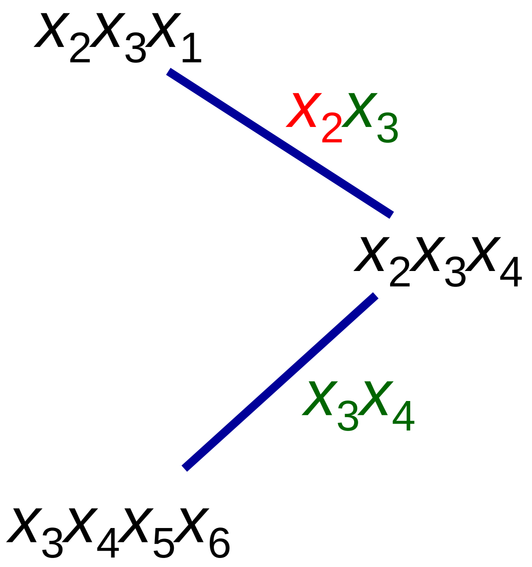
Designing the Nonserial BDD

$x_2 \; x_3 \; x_4 \; x_1 \; x_5 \; x_6$

BDD design

x_2

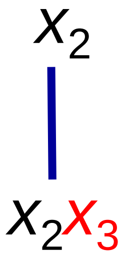
Join tree



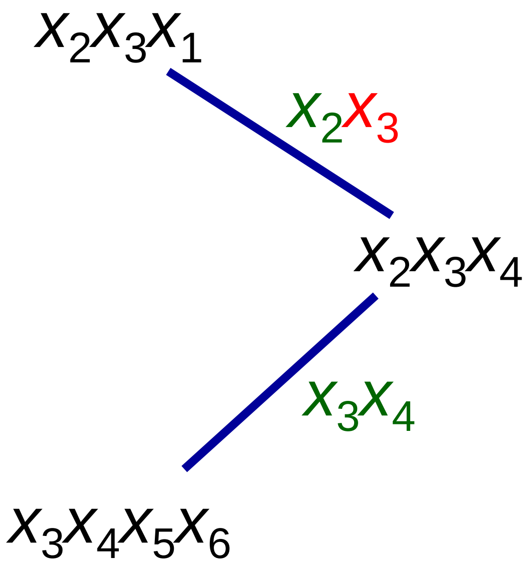
Designing the Nonserial BDD

$x_2 \ x_3 \ x_4 \ x_1 \ x_5 \ x_6$

BDD design



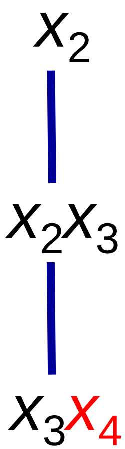
Join tree



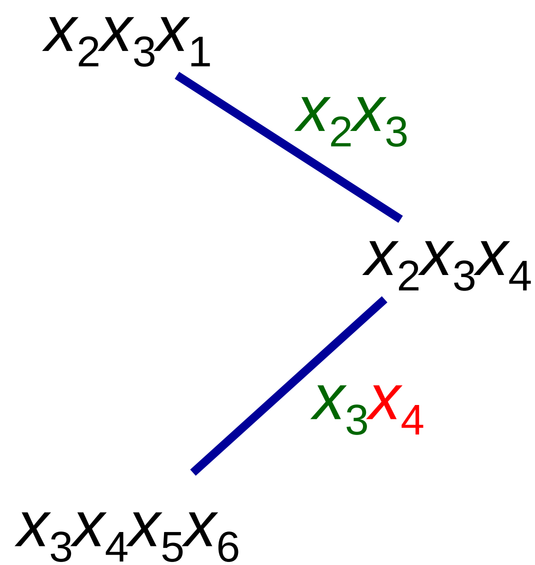
Designing the Nonserial BDD

$x_2\ x_3\ x_4\ x_1\ x_5\ x_6$

BDD design



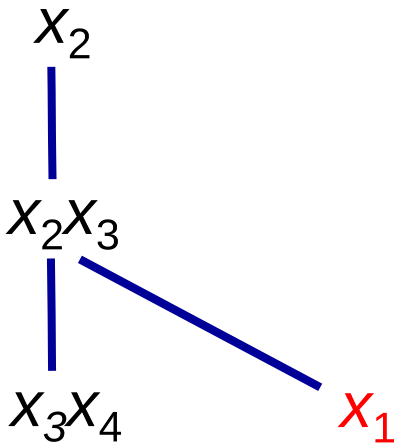
Join tree



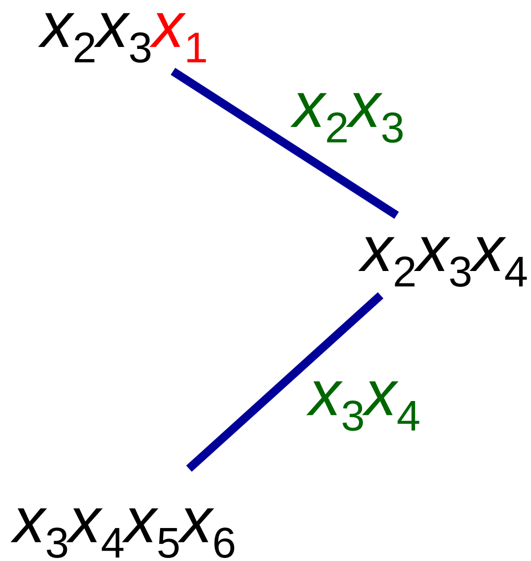
Designing the Nonserial BDD

$x_2 \ x_3 \ x_4 \ x_1 \ x_5 \ x_6$

BDD design



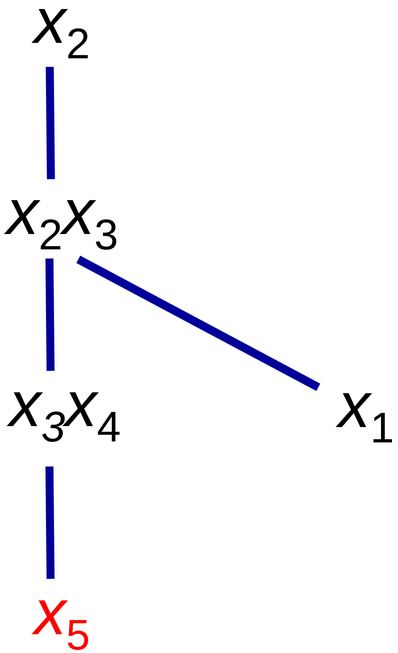
Join tree



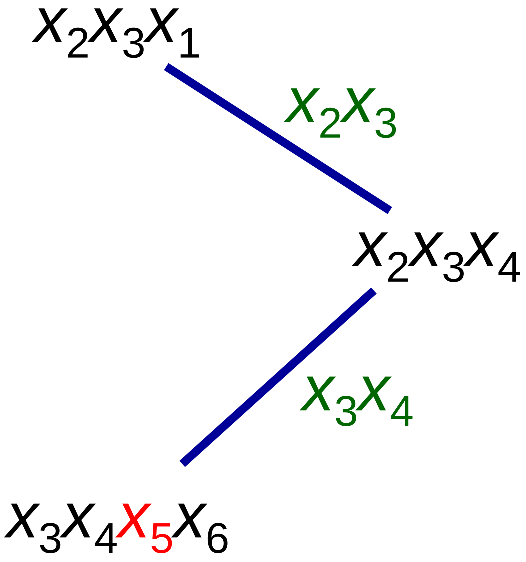
Designing the Nonserial BDD

$x_2\ x_3\ x_4\ x_1\ x_5\ x_6$

BDD design

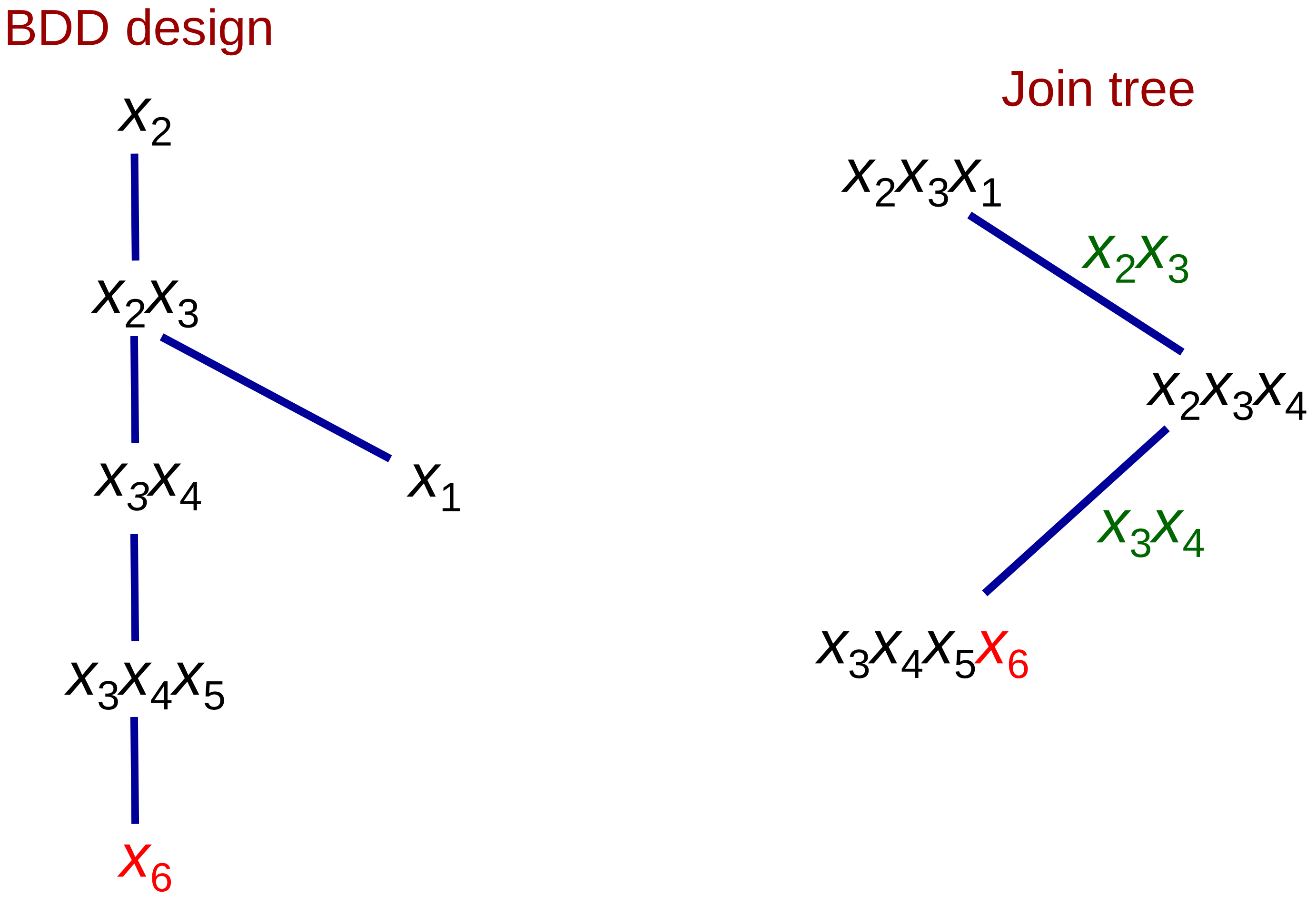


Join tree



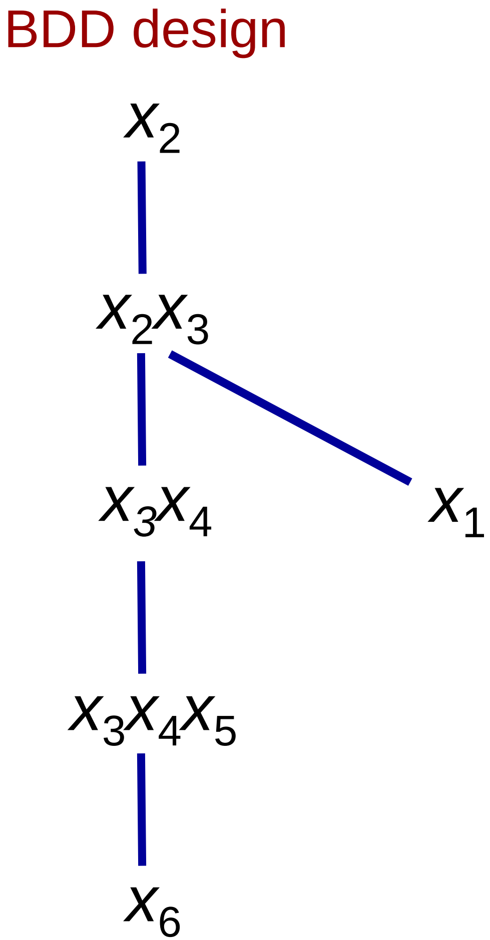
Designing the Nonserial BDD

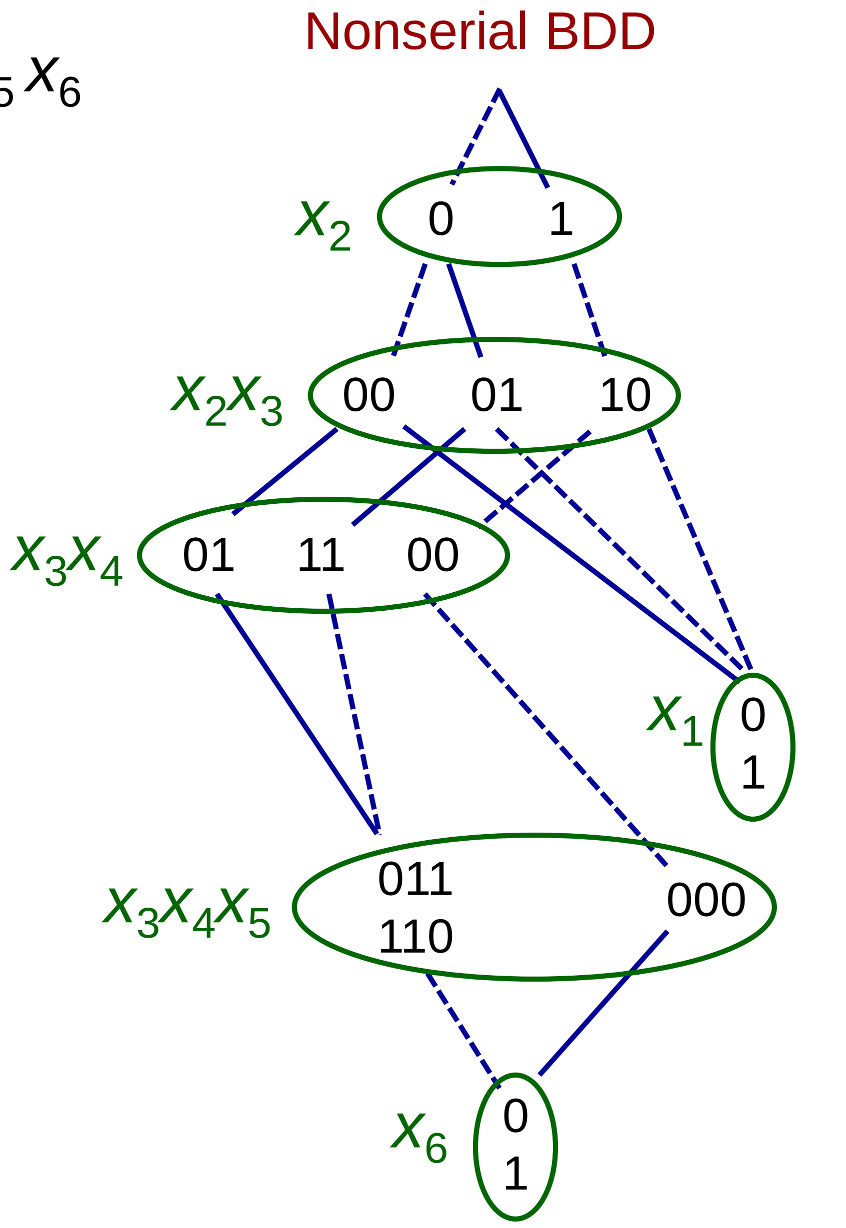
$x_2\ x_3\ x_4\ x_1\ x_5\ x_6$



Designing the Nonserial BDD

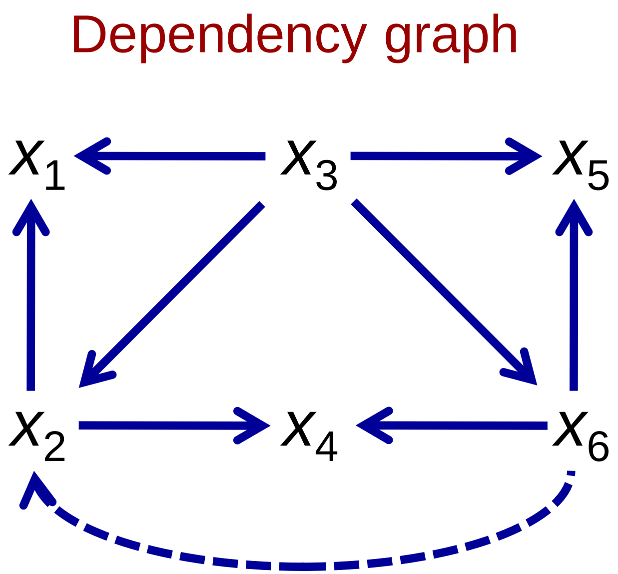
$x_2\ x_3\ x_4\ x_1\ x_5\ x_6$



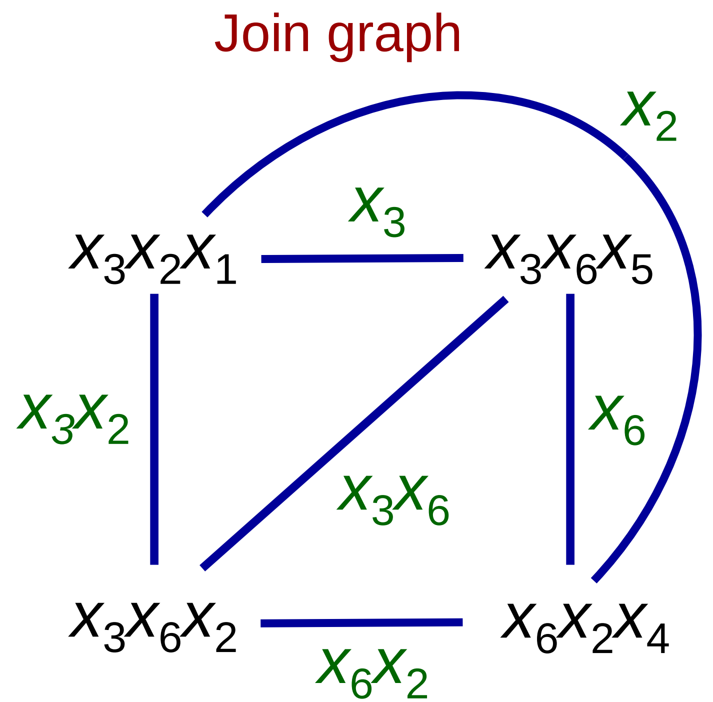


Another Variable Ordering

$x_3\ x_6\ x_2\ x_5\ x_1\ x_4$



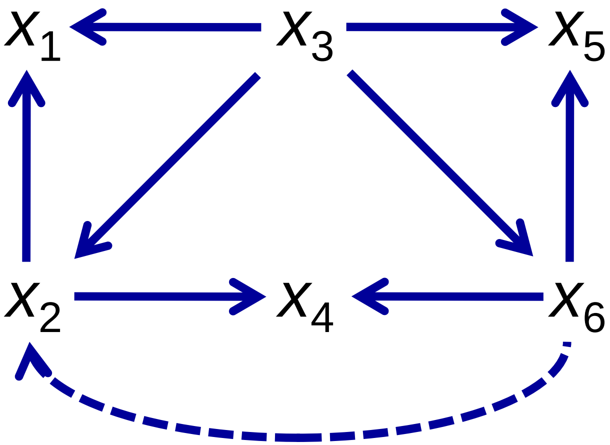
Induced width = 2



Constructing the Join Tree

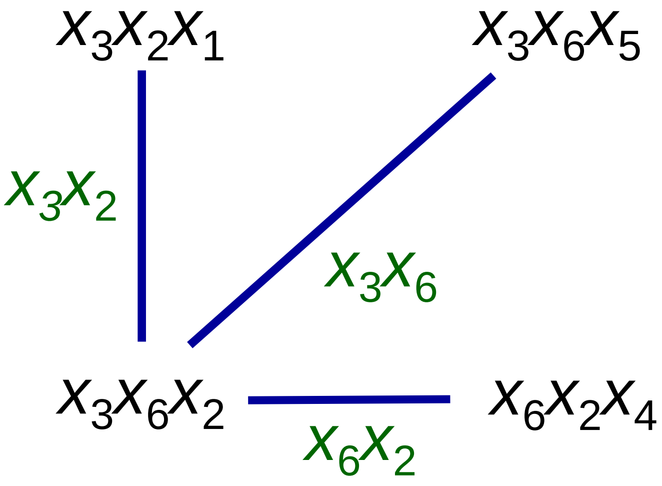
$x_3 \ x_6 \ x_2 \ x_5 \ x_1 \ x_4$

Dependency graph



Induced width = 2

Join tree



Tree width = 2

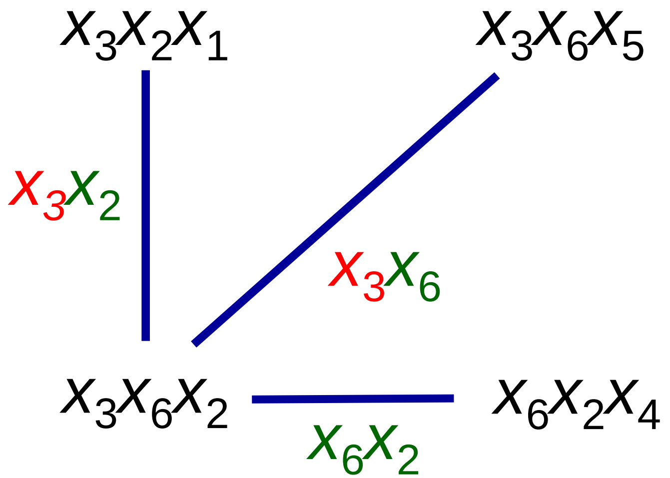
Designing the BDD

$x_3\ x_6\ x_2\ x_5\ x_1\ x_4$

BDD design

x_3

Join tree

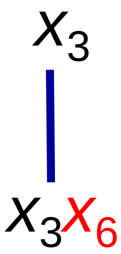


Tree width = 2

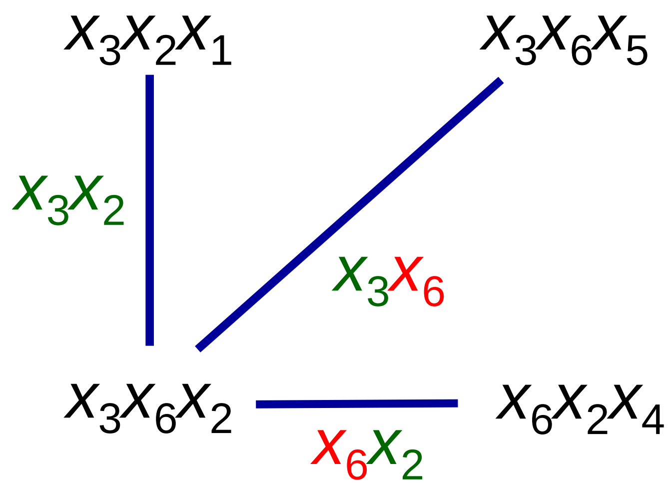
Designing the BDD

$x_3 \ x_6 \ x_2 \ x_5 \ x_1 \ x_4$

BDD design



Join tree

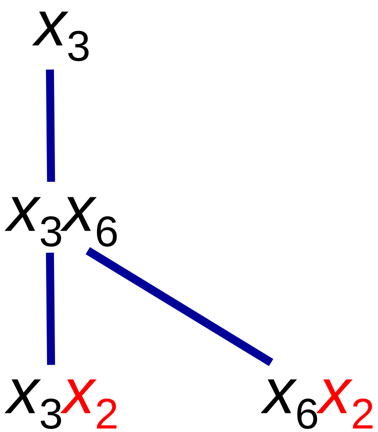


Tree width = 2

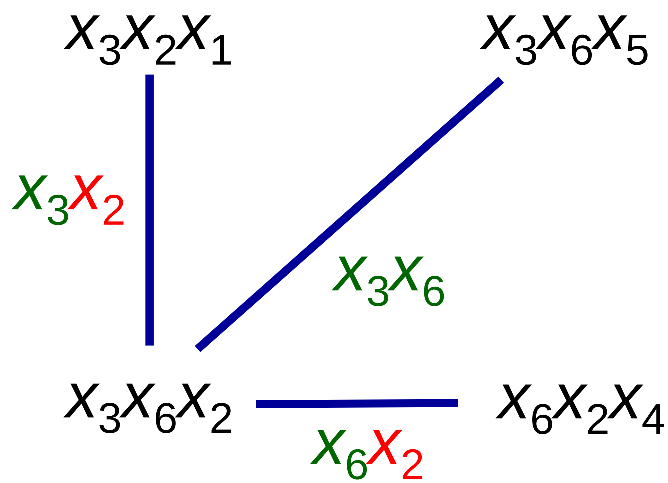
Designing the BDD

$x_3 \; x_6 \; x_2 \; x_5 \; x_1 \; x_4$

BDD design



Join tree

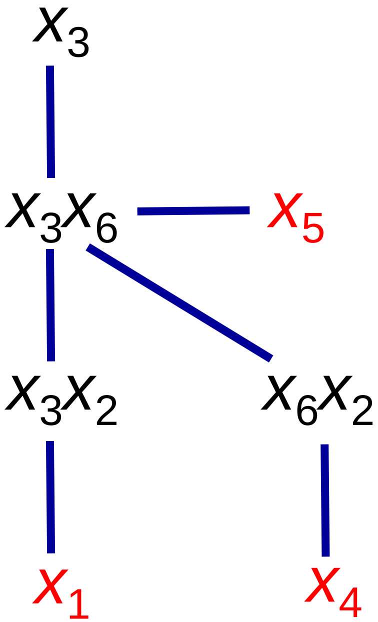


Tree width = 2

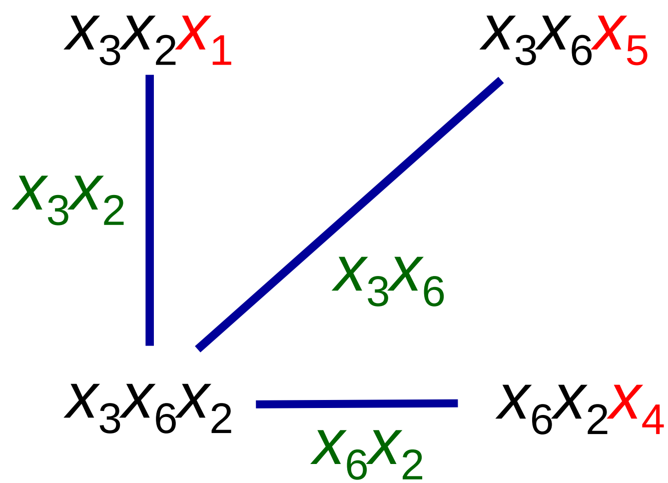
Designing the BDD

$x_3 \ x_6 \ x_2 \ x_5 \ x_1 \ x_4$

BDD design



Join tree

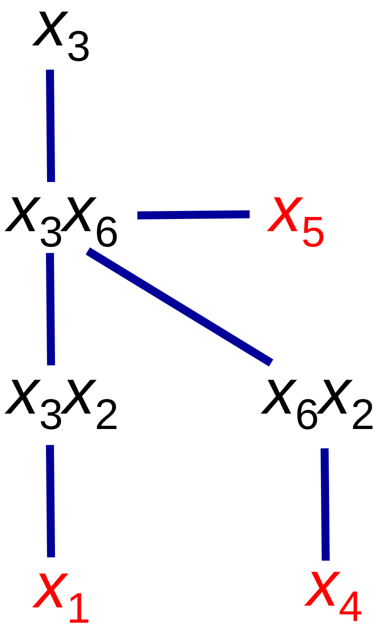


Tree width = 2

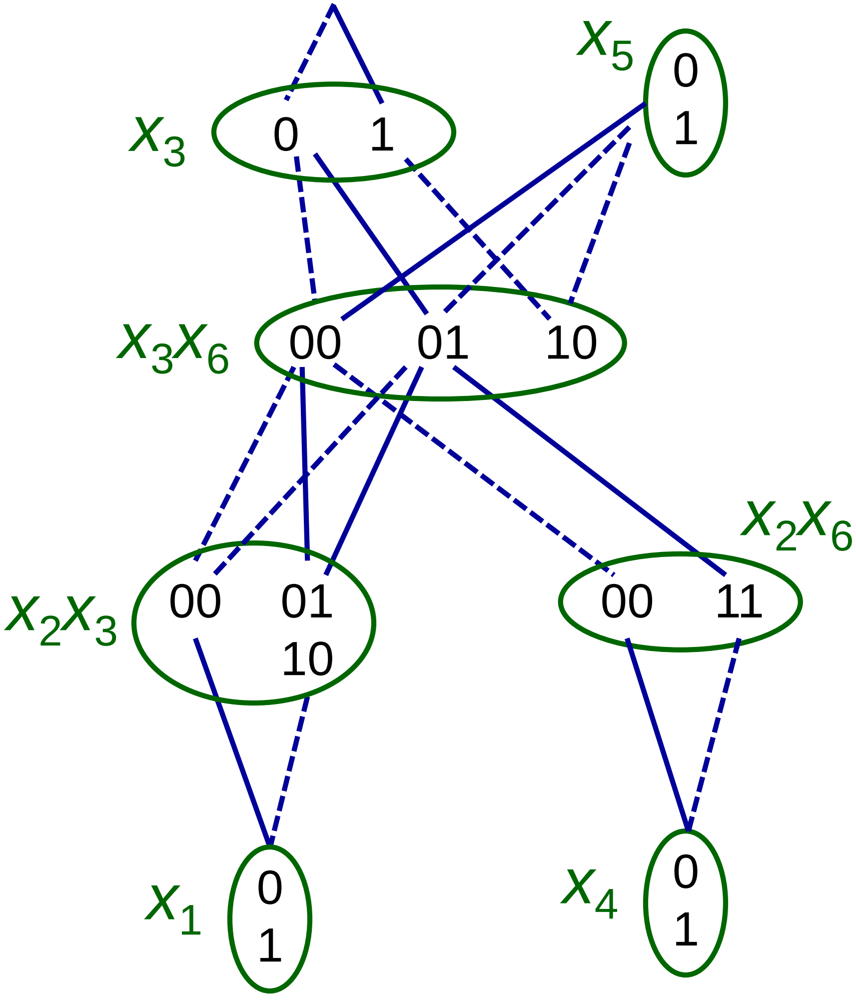
Nonserial BDD

$x_3\ x_6\ x_2\ x_5\ x_1\ x_4$

BDD design



Nonserial BDD



Current Research

- Broader applicability
  - Stochastic dynamic programming
  - Continuous global optimization
- Combination with other techniques
  - Lagrangean relaxation.
  - Column generation
  - Logic-based Benders decomposition
    - Solve separation problem

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